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ON DEFORMATION AND STRENGTH*#

C. C. Hsiao+ and C. S. Ting+

The mechanical strength of a medium is formulated in terms of microscopic behavior in the continuum. Both deformation and orientation of the microscopic elements are considered in the formulation. It is shown that under certain conditions the representation of the complicated time-dependent macro-modulus functions can be greatly simplified through the use of micro-behavior. The time-dependent macroscopic fracture strength is also obtained using microscopic considerations. The fracture strength is found to be linearly related with logarithm of time only for relatively large loads.

INTRODUCTION AND MATHEMATICAL MODEL

The macroscopic mechanical behavior of a material body is complicated as it is intimately affected by deformation and associated structural changes. Any investigation without taking these microscopic structural considerations is unlikely to give basic information or clear understanding. This is particularly true for materials having pronounced time-dependent properties. Viscoelastic behavior and ultimate strength are such properties that cannot be accurately determined unless deformation is considered. The present report attempts to describe methods of analyzing the macroscopic viscoelastic behavior and the time-dependent mechanical strength of a medium as a consequence of deformation and orientation of microscopic elements in the medium.

The mathematical model which will be used for analysis is a matrix of oriented elements embedded in an arbitrary domain. The whole system can be either suspension of individual elements or a network of elements connected by flexible joints or both. Depending upon the nature of the molecular constitutions, the basic elements may be real or abstract molecular forces linear

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+ University of Minnesota, Minneapolis, Minnesota

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or nonlinear in their load deformation characteristics. In the vicinity of a point, deformation and orientation of parallel elements in the micro-continuum may be represented by motions of linear vectors. For small deformations the complete matrix will have little orientation effect. Considerations of deformations of individual elements will yield representative information on the macro-behavior of the system. However, when deformations become appreciable, the orientation of the micro-elements must be taken into consideration if significant information is to be obtained.

Referring to an arbitrary reference frame, the deformation of any element can be easily expressed. Denoting the instantaneous direction of an element by a unit vector s_i then under load the deformation of the elements can be expressed as $r_o\varepsilon_{mn}s_m s_n$ where r_o is the original length of the element and ε_{mn} is the strain tensor. In general, the components of ε_{mn} are functions of time t and space variables. For small deformations there is only negligible amount of orientation and $s_m s_n$ can be regarded as independent of time. When deformation becomes significant, orientation of the elements must be considered. This can be accomplished by introducing a density function of orientation $\rho(\varepsilon)$ in terms of large strain ε. As mentioned earlier, depending upon the nature of molecular constitution two extreme cases or their combinations may be considered. Using orthogonal spherical coordinates (r,θ,ϕ) consider the case for the deformation of a unit sphere with randomly oriented elements to a spheroid symmetrical about its axis, the density of the probability distribution function of orientation of the elements is[1]

$$\rho(\varepsilon) = \rho(o)\,\frac{(1+\varepsilon)^3}{[\cos^2\theta + (1+\varepsilon)^3\sin^2\theta]^{3/2}} \tag{1}$$

where $\rho(o) = \frac{1}{4\pi}$ is a constant representing random distribution density function. However, if these elements are connected together by flexible joints, then[2]

$$\rho(\varepsilon) = \rho(o)\,\frac{\alpha}{[\cos^2\frac{\theta}{2} + \alpha\,\sin^2\frac{\theta}{2}]^2} \tag{2}$$

where α and ε are related in the following form

$$\varepsilon = 1 + \frac{\alpha}{\alpha^2 - 1}\ln\frac{\alpha+1}{2\alpha} \tag{3}$$

These distribution functions are obtained for continuous systems. For discrete systems, modifications can be easily introduced.[3] Therefore, it seems that for any kind of media the general formulation should apply equally as well.

MACROSCOPIC AND MICROSCOPIC BEHAVIOR

In order to obtain macroscopic information incorporating microscopic behavior, consider the stress tensor σ_{ij} in the vicinity of a point where at least an element is attached. If $\lambda\rho$ is the number of elements per unit volume oriented in a particular direction, $f(\theta,\phi,t)$ the fraction of unbroken elements, a function of orientation and time t, and $F = \psi A$ is the force exerted on each element of cross-sectional area A and length r_o corresponding to a stress ψ per element, then the traction force contribution by these elements can be found as[1]

$$F(\theta,\phi)\lambda r_o\rho(\theta,\phi)f(\theta,\phi,t)s_i s_j d\omega \tag{4}$$

where $d\omega$ is the solid angle. Since $\lambda r_o A = 1$, therefore considering all the possible orientations, the stress tensor can be expressed as follows

$$\sigma_{ij} = \int_\omega \psi\rho f s_i s_j d\omega \tag{5}$$

Now let us consider the one-dimensional constitutive equation of the element. In general the stress is expressible as a functional of deformation history subject to restrictions imposed upon by principles of objectivity. Under suitable conditions, for small finite deformations ε, ψ can be given as

$$\psi = \int_{-\infty}^{t} {}_1E(t-\tau_1)\dot{\varepsilon}(\tau_1)d\tau_1 + \int_{-\infty}^{t}\int_{-\infty}^{t} {}_2E(t-\tau_1,\ t-\tau_2)\dot{\varepsilon}(\tau_1)\dot{\varepsilon}(\tau_2)d\tau_1 d\tau_2 + \int_{-\infty}^{t}\int_{-\infty}^{t}\int_{-\infty}^{t} {}_3E(t-\tau_1,t-\tau_2,t-\tau_3)\dot{\varepsilon}(\tau_1)\dot{\varepsilon}(\tau_2)\dot{\varepsilon}(\tau_3)d\tau_1 d\tau_2 d\tau_3 \tag{6}$$

where kernel functions, ${}_1E$, ${}_2E$, ${}_3E$ depend upon the micro-behavior of the element material. Supposing that the elements are initially undisturbed, the stress components then can be put in the form

$$\sigma_{ij} = \int_\omega \rho(\theta,\phi)f(t)s_i s_j \Big[\int_o^t {}_1E(t-\tau_1)\overline{\dot{\varepsilon}_{mn}s_m s_n(\tau_1)}\, d\tau_1$$

$$+ \int_o^t\int_o^t {}_2E(t-\tau_1,t-\tau_2)\, \overline{\dot{\varepsilon}_{mn}s_m s_n(\tau_1)}\; \overline{\dot{\varepsilon}_{pq}s_p s_q(\tau_2)}\, d\tau_1 d\tau_2$$

$$+ \int_o^t\int_o^t\int_o^t {}_3E(t-\tau_1,t-\tau_2,t-\tau_3)\overline{\dot{\varepsilon}_{mn}s_m s_n(\tau_1)}\; \varepsilon_{pq}s_p s_q(\tau_2)$$

$$\overline{\dot{\varepsilon}_{rs}\varepsilon_r\varepsilon_s(\tau_3)}\, d\tau_1 d\tau_2 d\tau_3 + \ldots\ldots\Big] d\omega \qquad (7)$$

σ_{ij} is also the macroscopic stress field for a homogeneous material body as the average stress field is the same as that at any point. On the basis of somewhat comparable finite small deformation considerations, the same stress tensor can also be represented in terms of macroscopic relaxation modulus functions G_{ijmn}, etc. as follows

$$\sigma_{ij}(t) = \int_o^t {}_1G_{ijmn}(t-\tau_1)\dot{\varepsilon}_{mn}(\tau_1)d\tau_1$$

$$+ \int_o^t\int_o^t {}_2G_{ijmnpq}(t-\tau_1,t-\tau_2)\dot{\varepsilon}_{mn}(\tau_1)\dot{\varepsilon}_{pq}(\tau_2)d\tau_1 d\tau_2$$

$$+ \int_o^t\int_o^t\int_o^t {}_3G_{ijmnpqrs}(t-\tau_1,t-\tau_2,t-\tau_3)\dot{\varepsilon}_{mn}(\tau_1)\dot{\varepsilon}_{pq}(\tau_2)\cdot$$

$$\dot{\varepsilon}_{rs}(\tau_3)d\tau_1 d\tau_2 d\tau_3 \qquad (8)$$

For small strains, $s_m s_n$ will be independent of molecular orientation, and the linear macroscopic behavior can be related to the microscopic behavior of elements in an oriented system in the following manner:

$$\int_o^t G_{ijmn}(t-\tau)\varepsilon_{mn}(\tau)d\tau = \int_\omega \rho f s_i s_j s_m s_n \int_o^t E(t-\tau)\varepsilon_{mn}(\tau)d\tau d\omega \qquad (9)$$

For the convenience of illustration, if f is considered to be independent of time, then $G_{ijmn}(t)$ can be related to $E(t)$ indicating that a single function of a microscopic element governs the macroscopic time-dependent behavior and the variation of the orientation of the elements. Differentiating (9) with respect to time t, we obtain:

$$\begin{aligned}
G_{1111}(t) &= \int_\omega \rho f s_1 s_1 s_1 s_1 d\omega E(t)\\
G_{1122}(t) &= \int_\omega \rho f s_1 s_1 s_2 s_2 d\omega E(t)\\
G_{1133}(t) &= \int_\omega \rho f s_1 s_1 s_3 s_3 d\omega E(t)\\
G_{2222}(t) &= \int_\omega \rho f s_2 s_2 s_2 s_2 d\omega E(t)\\
G_{2233}(t) &= \int_\omega \rho f s_2 s_2 s_3 s_3 d\omega E(t)\\
G_{3333}(t) &= \int \rho f s_3 s_3 s_3 s_3 d\omega E(t)
\end{aligned} \qquad (10)$$

Referring to a spherical coordinate system and letting the direction of any representative element be designated by the angles θ and ϕ in the usual manner, then the joint probability distribution function is

$$\int_o^\phi\int_o^\theta \frac{1}{2\pi}\sin\theta d\theta d\phi \qquad \text{for} \qquad \begin{matrix} 0 \le \theta \le \frac{\pi}{2} \\ 0 \le \phi \le 2\pi \end{matrix} \qquad (11)$$

and $s_i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. For the case $\rho(\varepsilon)$ is represented in (1), we find

$$G_{1111}(t) = \frac{3\rho(0)f}{8}\left\{1 - \frac{2}{K^2}\left[1 - (1-K^2)^{1/2}\frac{1}{K}\sin^{-1}K\right] + \frac{1}{2K^4}\left[3 - K^2(1-K^2)^{1/2}\frac{1}{K}\sin^{-1}K\right]\right\}E(t)$$

$$G_{1122}(t) = \frac{1}{3}G_{1111}(t)$$

$$G_{1133}(t) = G_{2233}(t) = \frac{\rho(0)f}{2}\left\{\frac{1}{K^2}\left[1 - (1-K^2)^{1/2}\frac{1}{K}\sin^{-1}K\right] - \frac{1}{2K^4}\left[3 - K^2 - 3(1-K^2)^{1/2}\frac{1}{K}\sin^{-1}K\right]\right\}E(t)$$

$$G_{2222}(t) = G_{1111}(t)$$

and

$$G_{3333}(t) = \rho(0)f\left\{\frac{1}{2K^4}\left|3 - K^2 - 3(1-K^2)^{1/2}\frac{1}{K}\sin^{-1}K\right|\right\}E(t) \tag{12}$$

where

$$K = 1 - (1+\varepsilon)^{-3}$$

It is seen that for this simple case all macro-relaxation functions are possessing the same functional form as the micro-relaxation function E(t). The advantage of using these relationships in obtaining viscoelastic information for anisotropic bodies is evident.

It should be pointed out that f is in general a time-dependent function which can be evaluated from the statistical theory of the absolute reaction rate[1]

$$\frac{df}{dt} = K_r\left(\frac{1}{4\pi} - f\right) - K_b f \tag{13}$$

where $K_r = \omega_r e^{-(U/RT + \gamma\psi)}$ is the rate of reformation of broken elements and $K_b = \omega_b e^{-(U/RT-\beta\psi)}$, the rate of rupturing of unbroken elements. ω_r and ω_b are respectively frequencies of motion associated with either reformation or breakage processes. U is activation energy, R is a universal constant, T is absolute temperature and β and γ are positive constants. Once the functional form of f is determined, the relaxation modulus functions can then be properly modified. Furthermore, through the correlation of the viscoelastic constitutive equations, it may also be possible to deduce the creep functions of microscopic elements of the medium as well.

TIME-DEPENDENT MACRO- AND MICRO-STRENGTH

The time-dependent fracture of any medium can be studied by solving (13), from which

$$f = \frac{1}{4\pi} e^{-\int_o^t (K_r+K_b)dt}\left[\int_o^t K_r e^{\int_o^t (K_r+K_b)dt} dt + \frac{1}{2}\right] \tag{14}$$

According to the absolute reaction rate theory, the effect of stress on elastic deformation and viscous flow is capable of being related to a comprehensible activation process as a conse-

quence of the movement of the whole or a segment of a molecule, from one equilibrium state to the next. If U is the original potential energy barrier to be crossed between two equilibrium states, the frequency with which such steps occur under the influence of the thermal agitation will be proportional to $e^{-U/RT}$. After a stress ψ is applied to the elements on the system, the energy barrier becomes modified to $U/RT-\beta\psi$ in the direction of the applied stress, to $U/RT+\gamma\psi$ in the opposite sense where, as stated earlier, β and γ are constants depending upon the modification of the energy barrier. As a result both the rates of modified kinetic process become different exponential functions of stress. Since $K_b = \omega_b e^{-(U/RT-\beta\psi)}$ and $K_r = \omega_r e^{-(U/RT+\gamma\psi)}$ for a large value of stress $\psi(t)$, K_r will be very much smaller compared with K_b. To a first approximation, (14) may be reduced to

$$f = \frac{1}{8\pi} e^{-\int_o^t \omega_b e^{-(U/RT-\beta\psi)}d\tau} \tag{15}$$

It is generally recognized that the molecular orientation processes will take place before the inception of fracture[4]. Assuming that all the elements will be fully oriented before fracture, then $\rho = 1$ and the stress function $\psi(t)$ in each element will be given as

$$\psi(t) = \frac{\sigma(t)}{f(t)} = 8\pi\sigma\, e^{\omega_b e^{-U/RT}\left[\int_o^t e^{\beta\psi(\tau)}d\tau\right]} \tag{16}$$

Furthermore, U is a material constant and $\psi(t)$ may be considered as a continuous and multi-differentiable function of time, we may write, using (15) and (16)

$$\psi(t) = 8\pi\sigma_m e^{\omega_b e^{-U/RT}\left[\int_o^{t_m} e^{\beta[\psi(0)+\psi'(0)\tau+\frac{1}{2}\psi''(0)\tau^2+\cdots\cdots]}d\tau\right]} \tag{17}$$

where t_m is the time-to-fracture for a constant stress σ_m.

It seems quite reasonable, at this stage, to assume that the fracture strength is associated with a limiting value ψ_m beyond which every element oriented in the direction of applied stress will break. Then

$$\ell n \frac{8\pi\sigma_m}{\psi_m} + \omega_b e^{-U/RT}\left\{\int_o^{t_m} e^{\beta[\psi(0)+\psi'(0)\tau+\frac{1}{2}\psi''(0)\tau^2+..]}d\tau\right\} \qquad (18)$$

Again for simplicity by dropping higher order terms the first two terms are taken in evaluating the integral, we obtain

$$\ell n\frac{8\pi\sigma_m}{\psi_m} + \omega_b e^{-U/RT}\left\{\frac{e^{\beta\psi(0)}}{\beta\psi'(0)}\left[e^{\psi'(0)t_m} - 1\right]\right\} = 0 \qquad (19)$$

From (17) it is easily seen that $\psi(0) = 8\pi\delta(0) = \sigma_m$. Substituting in (19) and approximate the last terms, we have

$$\ell n \frac{8\pi\sigma_m}{\psi_m} + \omega_b e^{-U/RT} e^{\beta 8\pi\sigma_m} e^{\ell n t_m} = 0 \qquad (20)$$

For relatively large values of σ_m, $e^{\beta 8\pi\sigma_m} \gg \ell n \frac{8\pi\sigma_m}{\psi_m}$ (20) can be approximated to show a linear relation between the fracture stress δ_m and logarithm t_m,

$$\sigma_m \cong \frac{1}{8\pi\beta}(U/RT - \ell n\, \omega_b t_m). \qquad (21)$$

Fig. 1 shows schematically the variation of the fracture strength as a function of logarithm of time represented by (20). This theoretical curve appears to fit very well with various findings[5]. In addition, for relatively large applied stresses (21) gives the approximate relationship which agrees fairly well with available experimental data[6].

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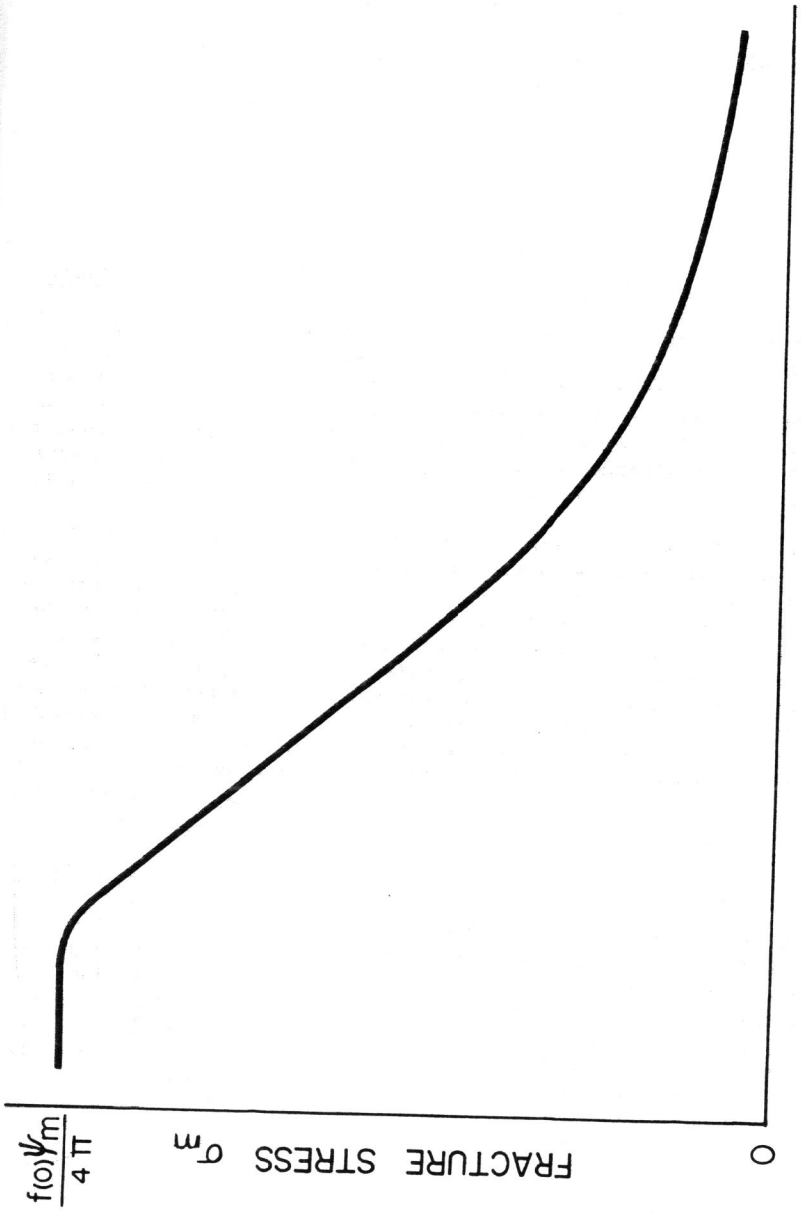

Fig. 1 Schematic representation of stress-lifetime relationship