A-22 FLEXURAL PROBLEMS OF CRACKS IN MIXED MEDIA[1]

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Abstract

The paper constitutes an extension of the Hilbert boundary problem, originally developed by Muskhelishvili for solving plane problems with cuts or cracks, to the transverse flexure of thin plates of dissimilar (or similar) materials bonded along straightline segments. The unbonded portion of the interface may be regarded as crack-like imperfections. Using the properties of Plemelj formulae and Cauchy integrals, sectionally holomorphic functions are developed for one or more cracks distributed along the dividing line of two dissimilar materials under flexure.

The flexural stress in the vicinity of a crack between two different materials is found to be of an oscillating character with singularity of the order of $r^{-1/2}$, r being the radial distance from the crack point. This indicates that interpenetration of certain parts of the crack boundary may take place, a physically impossible condition. Crack systems of interest such as concentrated couples applied to the surfaces of a finite crack and an infinite series of collinear line cracks in joined materials are also examined in detail. Presumably, the current fracture mechanics theories of cracks in a homogeneous material may be extended to the prediction of remaining strength of bonded dissimilar materials with cracks along the interface.

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Introduction

Many important problems of the theory of elasticity may be solved by reduction to the problem of linear relationship or Hilbert problem in complex function theory. In particular, when the region under consideration is the half-plane or the plane with cuts or cracks along one and the same straight line, the Hilbert formulation is an expedient as shown by Muskhelishvili [1][3] for solving extensional problems. The object of the present paper is to extend such a formulation to flexural problems of thin plates made of two dissimilar materials with lines of discontinuities along the common bond line.

Plane problems of fault lines at the interface of two different materials have been considered by several authors. Using the eigen-function-expansion technique, Williams [2] has investigated the problem of a semi-infinite crack between two joined half-planes. He discovered that the stresses in the immediate vicinity of the crack tip are infinitely large in magnitude and change the sign an infinite number of times. This oscillateory phenomenon was discussed at length by Salganik [3] based on the results of Cherepanov [4] who has formulated the problem of two half-planes bonded to each other along a finite number of straight-line segments by means of the problem of linear relationship. Salganik has further showed that the opposite sides of the crack may interfere in a region close to the crack tip. In other words, the crack surfaces may wrinkle and overlap, which is physically inadmissable. The same conclusion was obtained by England [5]. Recently, Rice and Sih [6] have combined the method of eigenfunction-expansion with complex function theory to solve a number of extensional problems of cracks in dissimilar media and found that in bi-material problems the stresses and rotations at infinity cannot be specified independently.

In another paper, Sih and Rice [7] have also considered briefly the flexure of a bi-material plate containing a semi-infinite crack. With the aid of eigenfunction expansions, the flexural stresses were found to oscillate rapidly near the crack tip, a phenomenon similar to that observed in plane problems of cracks. Results were expressed in terms of Goursat functions [1], but were not determined for specific loadings and crack geometries. In this paper, the non-homogeneous plate is assumed to have a finite or infinite number of collinear cracks placed along a straight line between two dissimilar (similar) materials. Boundary problems are formulated in terms of sectionally holomorphic functions following the problem of linear relationship. The method of solution involves Plemlj formulae and Cauchy integrals.

Considered in detail is the first fundamental problem of flexure of bi-material plates with cracks. Solution is obtained in general form such that it includes many special cases of fundamental interest. First, a solution is found explicitly for the problem of a single line crack in an infinite plate of dissimilar materials with uniform bending moments at infinity. The same configuration with concentrated couples applied to the crack surfaces is also examined. In order to study the interaction of cracks between two bonded materials, the case of a periodic array of collinear cracks is solved in closed form. In all of the aforementioned examples, wrinkling and overlapping of the crack surfaces occur in the displacements near the ends of the cracks. Such a physical impossibility, however, occurs only in those places near which the solution cannot be described satisfactorily by the linear theory of elasticity. The results of the present investigation are of interest in connection with the propagation of cracks associated with welded or bonded dissimilar materials under flexural loads.

[3] Numbers in brackets refer to References at end of paper.

Governing Equations of the Flexural Problem with Cracks

Let $z = x + iy$ be the complex variable of any point of the mid-surface of the plate made of two dissimilar elastic half-planes with lines of discontinuities along the bond. The upper half-plane S^+ is occupied by a material with elastic constants E_1 and ν_1, and the lower half-plane S^- by a material with elastic constants E_2 and ν_2. In the sequel, all quantities associated with region S^+ will carry the subscript 1, and all those pertaining to region S^- will be attached with subscript 2.

Both Lekhnitsky [8] and Savin [9] have applied the complex variable method of Muskhelishvili [1] to solve flexural problems of thin plates with holes using the Poisson-Kirchhoff theory. For isotropic plates of one material, their formulation requires the determination of two complex functions $\phi'(z)$ and $\psi'(z)$ of the complex variable $z = x + iy$. Denoting

$$\Phi(z) = \phi'(z) \quad , \qquad \Psi(z) = \psi'(z)$$

the complete solution to a bimaterial problem may be expressed in terms of four complex functions $\Phi_j(z)$, $\Psi_j(z)$, $j=1,2$. For crack problems, it is advantageous to further introduce the function [1]

$$\Omega_j(z) = \bar{\Phi}_j(z) + z\bar{\Phi}'_j(z) + \Psi_j(z) \ , \quad j = 1,2$$

Therefore, the bending, twisting moments, and shear forces at any point of the plate may be computed from the functions $\Phi_j(z)$, $\Omega_j(z)$ as follows:

$$(M_x)_j + (M_y)_j = -4D_j(1+\nu_j)\mathrm{Re}[\Phi_j(z)] \tag{1}$$

$$(M_y)_j - (M_x)_j + 2i(H_{xy})_j = 2D_j(1-\nu_j)[\bar{\Omega}_j(z) - \Phi_j(z) - (z-\bar{z})\Phi'_j(z)] \tag{2}$$

$$(Q_x)_j - i(Q_y)_j = -4D_j\Phi'_j(z) \tag{3}$$

Similarly, it is convenient to express the out-of-plane deflection w_j and the in-plane displacements u_j, v_j in terms of two analytic functions $\phi(z)$, $\omega(z)$ of the complex variable $z = x + iy$

$$w_j = \text{Re}\left[\int \overline{\omega_j(z)}\, dz + \int \phi_j(z)\, dz - (z-\bar{z}) \int \phi_j'(z)\, dz\right] \qquad (4)$$

$$u_j + i v_j = -\delta \left[\omega_j(\bar{z}) + \phi_j(z) + (z-\bar{z})\, \overline{\phi_j'(\bar{z})}\right] \qquad (5)$$

where δ is the thickness coordinate and

$$\Omega_j(z) = \omega_j'(z), \qquad j = 1, 2$$

The flexural rigidity of the plate is denoted by $D_j = E_j h_j^3/12(1-\nu_j^2)$ with E_j, ν_j and h_j, being the Young's modulus, Poisson's ratio, and the thickness of the plate, respectively.

It should be remarked that the representation of the moments, shear forces and displacements by means of the functions $\phi_j(z)$, $\omega_j(z)$ and $\Phi_j(z)$, $\Omega_j(z)$ is valid only for the plane with lines of discontinuities along segments of the real axis. Upon assuming that[4]

$$\lim_{y \to 0} y\, \Phi_j'(t+iy) = 0, \qquad j = 1, 2$$

and t do not coincide with the singular points of the lines of discontinuities or cracks, the last terms in eqs. (2) and (5) vanish as z approaches $\bar{z}$. Hence, by placing the cracks along the real axis, the rather special representation given by eqs. (1) to (5) simplifies the boundary conditions of the flexural problem considerably.

Within the frame work of small-deflection theory of thin elastic plates, the boundary conditions on the crack surfaces and the continuity conditions along the bond lines will be satisfied only in the sense of Kirchhoff. That is the three conditions prescribing $(M_y)_j$, $(H_{xy})_j$ and $(Q_y)_j$ can be replaced by two conditions as

$$m_j(t) = (M_y)_j, \quad f_j(t) = \int_0^t (Q_y)_j\, dt + (H_{xy})_j, \quad j = 1, 2 \qquad (6)$$

where $m_j(t)$ and $f_j(t)$ are the bending and equivalent twisting moments

[4] Throughout this paper, z assumes the value of t on the real axis.

per unit length of the plate boundary, respectively. In general, m_j and f_j are expressible in terms of the functions $\Phi_j(z)$ and $\Omega_j(z)$:

$$m_j + i f_j = D_j(1-\nu_j)\left[\Omega_j(\bar{z}) - \mu_j \Phi_j(z) + (z-\bar{z})\overline{\Phi_j'(\bar{z})}\right] \qquad (7)$$

where

$$\mu_j = \frac{3+\nu_j}{1-\nu_j}, \quad j = 1, 2 \qquad (8)$$

In addition, since the functions $\phi(z)$, $\omega(z)$ in eq. (5) may be multivalued, the flexural problem of cracks will be formulated in terms of u_j', v_j', i.e., the partial derivatives $\partial u_j/\partial x$, $\partial v_j/\partial x$. Differentiating eq. (5) with respect to x gives

$$u_j' + i v_j' = -\delta\left[\Omega_j(\bar{z}) + \Phi_j(z) + (z-\bar{z})\overline{\Phi_j'(\bar{z})}\right] \qquad (9)$$

Thus, the required solution of the flexural plate problem under consideration lies in the determination of the sectionally holomorphic functions $\Phi_j(z)$, $\Omega_j(z)$, $j = 1, 2$ as defined by Muskhelishvili [1]. For large $|z|$, these functions take the forms [9]

$$\Phi_j(z) = \Gamma_j + \frac{1}{2\pi i}\, \frac{M_x^* + iM_y^*}{4D_j}\, \frac{1}{z} + 0\left(\frac{1}{z^2}\right) \qquad (10)$$

$$\Omega_j(z) = \Gamma_j + \Gamma_j^* + \frac{1}{2\pi i}\, \frac{M_x^* + iM_y^*}{4D_j}\, \frac{1}{z} + 0\left(\frac{1}{z^2}\right) \qquad (11)$$

where $M_x^* + iM_y^*$ is the resultant vector of the moments applied to the crack surfaces and Γ_j, Γ_j^* are related to $(M_x^\infty)_j$, $(M_y^\infty)_j$, the bending moments at infinity, i.e.,

$$\Gamma_j = -\frac{(M_x^\infty)_j + (M_y^\infty)_j}{4D_j(1+\nu_j)}, \qquad \Gamma_j^* = \frac{(M_y^\infty)_j - (M_x^\infty)_j}{2D_j(1-\nu_j)} \qquad (12)$$

Without affecting the general formulation of the problem, the twisting moment $(H_{xy}^\infty)_j$ at infinity is taken to be zero in this paper.

Hilbert Formulation of Flexural Problems of Cracks in Mixed Media

The mixed medium consists of two plates of dissimilar materials bonded along straight-line segments $L_j^*(j=1,2\ldots,n)$ of the real axis; the union of these segments is $L^* = L_1^* + L_2^* + \ldots + L_n^*$.

The remaining part of the real axis represents n number of cracks $L_j = a_j b_j (j=1,2,\ldots,n)$ such that their union is L and the ends are encountered in the order $a_1 b_1, a_2 b_2, \ldots\ a_n b_n$. Thus, $L^* \cup L$ is the entire x-axis.

If the subscripts + and - refer to the values of the functions on L approached from the regions S^+ and S^-, respectively, then the boundary conditions of the first fundamental problem may be written as

$$m_1^+ + i f_1^+ = -D_1(1-\nu_1)p^+(t), \text{ on } L$$
$$m_2^- + i f_2^- = -D_2(1-\nu_2)p^-(t), \text{ on } L \tag{13}$$

Continuity of moments and displacements across L^* requires that

$$m_1 + i f_1 = m_2 + i f_2, \text{ on } L^*$$
$$u_1' + i v_1' = u_2' + i v_2', \text{ on } L^* \tag{14}$$

Making use of eqs. (7) and (9), eqs. (13) and (14) become

$$\mu_1 \Phi_1^+(t) - \Omega_1^-(t) = p^+(t), \text{ on } L$$
$$\mu_2 \Phi_2^-(t) - \Omega_2^+(t) = p^-(t), \text{ on } L \tag{15}$$

and

$$\Phi_1(t) + \Omega_1(t) = \Phi_2(t) + \Omega_2(t), \text{ on } L^*$$
$$\gamma\left[\mu_1 \Phi_1(t) - \Omega_1(t)\right] = \mu_2 \Phi_2(t) - \Omega_2(t), \text{ on } L^* \tag{16}$$

in which γ stands for

$$\gamma = \frac{D_1(1-\nu_1)}{D_2(1-\nu_2)} \tag{17}$$

By means of analytic continuation, it can be shown from eqs. (16) that

the relationships

$$\Omega_1(z) = -\left(\frac{\gamma\mu_1+1}{1-\gamma}\right)\Phi_1(z) + \left(\frac{1+\mu_2}{1-\gamma}\right)\Phi_2(z)$$
$$\Omega_2(z) = -\gamma\left(\frac{1+\mu_1}{1-\gamma}\right)\Phi_1(z) + \left(\frac{\gamma+\mu_2}{1-\gamma}\right)\Phi_2(z) \tag{18}$$

will hold in the region $S^+ \cup S^-$ excluding the lines of discontinuities L. Therefore, once $\Phi_j(z)$, $j = 1, 2$ are known, $\Omega_j(z)$, $j = 1, 2$, follows from eqs. (18). Solving for $\Phi_j(z)$ in eqs. (15) and (16), there results

$$\left[\gamma\left(\frac{\gamma\mu_1+1}{\gamma+\mu_2}\right)\Phi_1(t)-\Phi_2(t)\right]^+ - \left[\gamma\left(\frac{\gamma\mu_1+1}{\gamma+\mu_2}\right)\Phi_1(t)-\Phi_2(t)\right]^- = \left(\frac{\gamma-1}{\gamma+\mu_2}\right)(\gamma p^+ - p^-), \tag{19}$$

and

$$\left[(1+\mu_1)\Phi_1(t)-(1+\mu_2)\Phi_2(t)\right]^+ + \frac{\mu_2}{\mu_1}\left(\frac{\gamma\mu_1+1}{\gamma+\mu_2}\right)\left[(1+\mu_1)\Phi_1(t)-(1+\mu_2)\Phi_2(t)\right]^-$$
$$= \frac{1-\gamma}{\mu_1(\gamma+\mu_2)}\left[\mu_2(1+\mu_1)p^+ + \mu_1(1+\mu_2)p^-\right] \tag{20}$$

Applying the Plemelj formula to eq. (19) yields

$$\gamma\left(\frac{\gamma\mu_1+1}{\gamma+\mu_2}\right)\Phi_1(z)-\Phi_2(z) = \left(\frac{1-\gamma}{\gamma+\mu_2}\right)\frac{1}{2\pi i}\int_L \frac{g(t)dt}{t-z} + \gamma\left(\frac{\gamma\mu_1+1}{\gamma+\mu_2}\right)\Gamma_1 - \Gamma_2 \tag{21}$$

Eq. (20) is in the form of the nonhomogeneous Hilbert equation whose solution is given by Muskhelishvili [1]:

$$(1+\mu_1)\Phi_1(z)-(1+\mu_2)\Phi_2(z) = \frac{1-\gamma}{2\pi i X(z)}\int_L \frac{X^+(t)f(t)dt}{t-z} + \frac{P_n(z)}{X(z)} \tag{22}$$

From eqs. (18), (21), and (22), the general solution to the flexural problem of cracks in mixed media is

$$\Phi_1(z) = \frac{-(1+\mu_2)F_3(z) + (\gamma+\mu_2)F_4(z)}{\mu_2(1+\mu_1)+\gamma\mu_1(1+\mu_2)}$$
$$\Phi_2(z) = \frac{-(1+\mu_1)F_3(z) + \gamma(\gamma\mu_1+1)F_4(z)}{\mu_2(1+\mu_1)+\gamma\mu_1(1+\mu_2)} \tag{23}$$

and

$$\Omega_1(z) = -\frac{\mu_1(1+\mu_2)F_3(z)+\mu_2(\gamma\mu_1+1)F_4(z)}{\mu_2(1+\mu_1)+\gamma\mu_1(1+\mu_2)}$$
$$\Omega_2(z) = -\frac{\mu_2(1+\mu_1)F_3(z)+\gamma\mu_1(\gamma+\mu_2)F_4(z)}{\mu_2(1+\mu_1)+\gamma\mu_1(1+\mu_2)} \tag{24}$$

The two unknown functions $F_j(z)$, $j = 3, 4$, in eqs. (23) and (24) are

$$F_3(z) = \frac{1}{2\pi i}\int_L \frac{g(t)dt}{t-z} + \frac{\gamma+\mu_2}{1-\gamma}[\gamma(\frac{\gamma\mu_1+1}{\gamma+\mu_2})\Gamma_1 - \Gamma_2] \quad (25)$$

$$F_4(z) = \frac{1}{2\pi i X(z)}\int_L \frac{X^+(t)f(t)dt}{t-z} + \frac{1}{1-\gamma}\frac{P_n(z)}{X(z)} \quad (26)$$

It will be assumed that the functions

$$g(t) = -(\gamma p^+ - p^-), \quad f(t) = \frac{\mu_2(1+\mu_1)p^+ + \mu_1(1+\mu_2)p^-}{\mu_1(\gamma+\mu_2)} \quad (27)$$

satisfy the Hölder condition on L. Moreover, the Plemelj function

$$X(z) = \prod_{j=1}^{n}(z-a_j)^{\frac{1}{2}-i\varkappa}(z-b_j)^{\frac{1}{2}+i\varkappa} \quad (28)$$

is single-valued in the bi-material plane, cut along L, and the branch of X(z) is taken such that

$$\lim_{z\to\infty} z^{-n}X(z) = 1$$

The bi-material constant

$$\varkappa = \frac{1}{2\pi}\log[\frac{\mu_2}{\mu_1}(\frac{\gamma\mu_1+1}{\gamma+\mu_2})] \quad (29)$$

agrees with eq. (12) in [7]. In view of eq. (28), $P_n(z)$ must be a polynomial of degree not greater than n:

$$P_n(z) = A_o z^n + A_1 z^{n-1} + \ldots + A_n \quad (30)$$

The problem is basically reduced to the determination of the (n+1) coefficients in eq. (30). The coefficient A_o may be obtained immediately by eliminating $F_3(\infty)$ in eqs. (23) or eqs. (24), which give

$$A_o = (1+\mu_1)\Gamma_1 - (1+\mu_2)\Gamma_2$$

or (31)

$$A_o = \frac{1-\gamma}{\gamma\mu_1-\mu_2}[\mu_2(1+\mu_1)(\Gamma_1+\Gamma_1^*) - \mu_1(1+\mu_2)(\Gamma_2+\Gamma_2^*)]$$

At first sight, there seems to be an inconsistency in the value of A_o. A closer examination of eqs. (23) or (24) reveals that the bending moment in the x-direction at infinity suffers a discontinuity across the bond line, i.e.,

$$(M_x^\infty)_2 = \frac{1}{\gamma}(\frac{1+\mu_1}{1+\mu_2})(\frac{1-\mu_2}{1-\mu_1})(M_x^\infty)_1 + \frac{(1-\gamma)[3-(\mu_1+\mu_2)+\mu_1\mu_2]+2(\gamma\mu_1-\mu_2)}{\gamma(1-\mu_1)(1+\mu_2)}M_y^\infty \quad (32)$$

Eq. (32) may be derived either by equating $F_4(\infty)$ in $\Phi_j(\infty)$ or in $\Omega_j(\infty)$, j = 1, 2. By virtue of this equation, it can be shown that the two expressions of A_o in eqs. (31) are indeed equivalent. The remaining coefficients $A_1, A_2, \ldots, A_n$ must be found from the condition of single-valuedness of the displacements.

Hitherto, the functions $\Phi_j(z)$, $\Omega_j(z)$ were derived on the basis that the derivatives of displacements are continuous across L^*, eqs. (14). Therefore, the displacements u_j, v_j, calculated from $\Phi_j(z)$, $\Omega_j(z)$, are determined up to some arbitrary constants, say c_j, which may be different on different cracks, $L_j = a_j b_j$. This means that

$$u_1^+(t) + iv_1^+(t) = u_2^-(t) + iv_2^-(t) + c_j, \quad \text{on } L_j \quad (33)$$

However, the physical problem requires u_j+iv_j to have definite limits as z approaches any one of the ends a_j, b_j:

$$u_1^+(a_j) + iv_1^+(a_j) = u_2^-(a_j) + iv_2^-(a_j)$$
$$u_1^+(b_j) + iv_1^+(b_j) = u_2^-(b_j) + iv_2^-(b_j) \quad (34)$$

These conditions may be satisfied if

$$c_1 = c_2 = \ldots c_n = 0$$

Furthermore, eqs. (34) may be subtracted and arranged in the form

$$[u_1^+(a_j)+iv_1^+(a_j)] - [u_1^+(b_j)+iv_1^+(b_j)] = [u_2^-(a_j)+iv_2^-(a_j)] - [u_2^-(b_j)+iv_2^-(b_j)]$$

Since u_j+iv_j are point functions, an equivalent statement of the above equation is

$$\int_{a_j}^{b_j}[u_1'^+(t) + iv_1'^+(t)]dt = \int_{a_j}^{b_j}[u_2'^-(t) + iv_2'^-(t)]dt \quad (35)$$

Using eq. (9), eq. (35) becomes

$$\int_{a_j}^{b_j}[\Phi_1^+(t) + \Omega_1^-(t)]dt = \int_{a_j}^{b_j}[\Phi_2^-(t) + \Omega_2^+(t)]dt \quad (36)$$

Eq. (36) may be further simplified if eqs. (23) and (24) are employed with the knowledge that $F_3^+(t) = F_3^-(t) = F_3(t)$. Hence, the single-valuedness of u_j+iv_j renders

$$\int_{a_j}^{b_j}[F_4^+(t) - F_4^-(t)]dt = 0, \quad j = 1, 2, \ldots, n \quad (37)$$

This is a system of n linear equations in the n unknowns $A_1, A_2, \ldots, A_n$. Note from eqs. (25), (26), and (27) that in the case $\Gamma_j = \Gamma_j^* = 0$ and $p^+ = p^- = 0$, the system, eqs. (37), can have no other solution except $A_1 = \ldots = A_n = 0$, which represents the trivial solution $\Phi_j(z) = \Omega_j(\bar{z}) = 0$. Therefore, eqs. (37) always have a unique solution as discussed by Muskhelishvili [1] for cracks under plane extension.

If the solution is to be entirely non-dislocational, the out-of-plane deflection w_j, $j = 1, 2$, of the bi-material plate must also be one-valued. Following the arguments used before and applying eq. (4), it is found that

$$\int_{a_j}^{b_j} \mathrm{Re}[\bar{\omega}_1^+(t) + \phi_1^+(t)]dt = \int_{a_j}^{b_j} \mathrm{Re}[\bar{\omega}_2^-(t) + \phi_2^-(t)]dt \ , \quad j = 1, 2, \ldots, n \qquad (38)$$

While eqs. (38) offer no additional information, they must be satisfied for non-dislocational problems of cracks subjected to flexural loads.

In order to describe the details of the method, specific examples will be considered in the sections to follow.

Cracked Plate of Dissimilar Materials Subjected to Uniform Moments

Suppose a bi-material plate with a single line crack on $y = 0$, $|x| \le a$ is stressed at infinity by bending moments $(M_x^\infty)_1$, $(M_y^\infty)_2$ and M_y^∞ in accordance with eq. (32). The edges of the crack are free from external loads so that $g(t) = f(t) = 0$. For a single crack, $n = 1$, eqs. (28) and (30) reduce to

$$X(z) = (z^2-a^2)^{\frac{1}{2}}\left(\frac{z-a}{z+a}\right)^{i\kappa} \ , \quad P_1(z) = A_o z + A_1 \qquad (39)$$

where

$$A_o = -\frac{(1-\gamma)(1+\mu_1)}{4D_1} M_y^\infty$$

The functions $F_j(z)$, $j = 3, 4$, are given by

$$F_3(z) = \frac{1}{1-\gamma}[\gamma(\gamma\mu_1+1)\Gamma_1 - (\gamma+\mu_2)\Gamma_2]$$
$$F_4(z) = \frac{1}{1-\gamma}\frac{A_o z + A_1}{X(z)} \qquad (40)$$

The only unknown A_1 in eqs. (40) can be obtained from eq. (37). It should be emphasized that the Plemelj function $X(z)$ may have different limits depending upon whether z is approached from the region S^+ or S^-. The two limiting expressions are related to each other by

$$X^-(t) = -\frac{\mu_2}{\mu_1}\left(\frac{\gamma\mu_1+1}{\gamma+\mu_2}\right)X^+(t) \qquad (41)$$

With the help of eqs. (40) and (41), eq. (37) becomes

$$\int_{-a}^{a} \left(\frac{t+a}{t-a}\right)^{i\kappa}\left[\frac{A_o t + A_1}{\sqrt{t^2-a^2}}\right]dt = 0 \qquad (42)$$

On replacing the line integral, eq. (42), by a contour integral, the result of integration requires

$$A_1 = -2ia\kappa A_o \qquad (43)$$

The obtained results may now be used to show that eq. (38) is satisfied.

Once the functions $F_j(z)$, $j = 3, 4$, are known, the flexural stresses in the plate may be computed from

$$(\sigma_x)_j = \frac{12\delta}{h_j^3}(M_x)_j \ , \quad (\sigma_y)_j = \frac{12\delta}{h_j^3}(M_y)_j \ , \quad (\tau_{xy})_j = -\frac{12\delta}{h_j^3}(H_{xy})_j$$
$$(\tau_{xz})_j = \frac{3(h_j^2-4\delta^2)}{2h_j^3}(Q_x)_j \ , \quad (\tau_{yz})_j = \frac{3(h_j^2-4\delta^2)}{2h_j^3}(Q_y)_j \qquad (44)$$

by way of eqs. (1), (2), (3), (23), and (24). Likewise, the deflections and displacements can be obtained in a straightforward manner.

Concentrated Couples on Crack Surfaces in Bonded Plates

Consider two dissimilar elastic plates bonded together along the x-axis except over the line crack, $-a \le x \le a$, which is opened by equal and opposite concentrated couples M applied to the crack surfaces at $x = b^+$ and $x = b^-$, where $b < a$. The bending moments at infinity are assumed to vanish. Thus, eqs. (12) and (31) imply $A_o = 0$. On the surface of the crack,

$$m_1^+ = m_2^- = M\delta(t-b) \ , \quad f_1^+ = f_2^- = 0 \ , \ \text{on } L \qquad (45)$$

where $\delta(t)$ is the Dirac delta function. Putting eqs. (45) into (13) and the subsequent result in eqs. (27) gives

$$g(t) = 0 \ , \quad f(t) = -\left[\frac{\mu_2(1+\mu_1)+\gamma\mu_1(1+\mu_2)}{\mu_1(\gamma+\mu_2)}\right]\frac{M\delta(t-b)}{D_1(1-\nu_1)} \qquad (46)$$

Since $\Gamma_1 = \Gamma_2 = 0$, eqs. (25) and (26) may be simplified:

$$F_3(z) = 0 \ , \quad F_4(z) = \frac{1}{2\pi i X(z)}\int_{-a}^{a}\frac{X^+(t)f(t)dt}{t-z} + \frac{1}{1-\gamma}\frac{A_1}{X(z)} \qquad (47)$$

Substituting eq. (46) into (47) and integrating, $F_4(z)$ becomes

$$F_4(z) = \frac{M}{2\pi i D_1(3+\nu_1)}\left[\frac{\mu_2(1+\mu_1)+\gamma\mu_1(1+\mu_2)}{\mu_1(\gamma+\mu_2)}\right]\frac{X(b)}{X(z)}\frac{1}{z-b} + \frac{1}{1-\gamma}\frac{A_1}{X(z)} \tag{48}$$

The structure of $X(z)$ for a single line crack is given by eqs. (39). There remains the evaluation of A_1. Making use of eqs. (41) and (48), eq. (37) takes the form

$$\int_{-a}^{a}\left(\frac{t+a}{t-a}\right)^{i\kappa}\left\{\frac{M}{2\pi i D_1(3+\nu_1)}\left[\frac{\mu_2(1+\mu_1)+\gamma\mu_1(1+\mu_2)}{\mu_1(\gamma+\mu_2)}\right]\frac{X(b)}{t-b} + \frac{A_1}{1-\gamma}\right\}\frac{dt}{\sqrt{t^2-a^2}} = 0 \tag{49}$$

whose solution demands $A_1 = 0$. For this problem, the single-valuedness of w_j, $j = 1, 2$, governed by eq. (38), is also satisfied. The complete solution may now be expressed by $\Phi_j(z)$, $\Omega_j(z)$ as follows:

$$\Phi_1(z) = \frac{M}{2\pi i D_1(3+\nu_1)}\frac{X(b)}{X(z)}\frac{1}{z-b}$$

$$\Phi_2(z) = \frac{M}{2\pi i D_1(3+\nu_1)}\left[\frac{\gamma(\gamma\mu_1+1)}{\gamma+\mu_2}\right]\frac{X(b)}{X(z)}\frac{1}{z-b}$$

and (50)

$$\Omega_1(z) = \frac{-M}{2\pi i D_1(3+\nu_1)}\left[\frac{\mu_2(\gamma\mu_1+1)}{\gamma+\mu_2}\right]\frac{X(b)}{X(z)}\frac{1}{z-b}$$

$$\Omega_2(z) = \frac{-M}{2\pi i D_1(3+\nu_1)}[\gamma\mu_1]\frac{X(b)}{X(z)}\frac{1}{z-b}$$

In passing, it is worthwhile noting that solutions to flexural problems with arbitrary bending moments $m(t)$ applied to the crack surfaces may be generated from eqs. (50), which represent the necessary Green's functions. As a simple example, setting $M = m\,dt$ and integrating over the segment from $-a$ to a, the solution for a bi-material plate with a crack opened by equal and opposite bending moments of uniform intensity m is obtained:

$$\Phi_1(z) = \frac{m}{D_1(3+\nu_1)}\left[\frac{\mu_1(\gamma+\mu_2)}{\mu_1(\gamma+\mu_2)+\mu_2(\gamma\mu_1+1)}\right]\left[\frac{z-2i\kappa a}{X(z)} - 1\right]$$

$$\Phi_2(z) = \left[\frac{\gamma(\gamma\mu_1+1)}{\gamma+\mu_2}\right]\Phi_1(z) \tag{51}$$

and

$$\Omega_1(z) = -\left[\frac{\mu_2(\gamma\mu_1+1)}{\gamma+\mu_2}\right]\Phi_1(z), \quad \Omega_2(z) = -[\gamma\mu_1]\Phi_1(z)$$

Alternatively, eqs. (51) may also be found directly from eqs. (23) to (26)

with $m_1^+ = m_2^- = m$, $f_1^+ = f_2^- = 0$, and $A_1 = 0$ ensuring the single-valuedness of the displacements.

Returning to the problem of concentrated couples, it should be remembered that eqs. (50) were derived from the statically-equivalent boundary conditions, eqs. (6), instead of the exact ones as mentinned earlier. Owing to such an approximation, the flexural stress distribution, computed from eqs. (50, in the immediate neighborhood of the crack point will naturally be affected. However, the modified boundary conditions do not alter the qualitative character of the flexural stress singularities. Keeping this in mind, the oscillatory behavior of the crack-tip flexural stresses and displacements will be studied.

It is of interest to compute the stress component σ_y along the interface $y = 0$, $|x| > a$. For brevity's sake, let $b = 0$. From eqs. (1), (2), (44), and (50), the bonding stress is found:

$$-(\sigma_y)_{y=0} = \frac{12M\delta}{h^3}\left(\frac{a}{t}\right)\frac{\cosh\pi\kappa}{\sqrt{t^2-a^2}}\cos\left[\kappa\log\left(\frac{t+a}{t-a}\right)\right], \quad |t| > a \tag{52}$$

Notice that both tensile and compressive stresses exist near the ends of the crack. This is mainly because the sign of $(\sigma_y)_{y=0}$ changes infinitely often as t approaches the values $-a$ and a. Similar oscillations have geen observed in [2-6] for the case of dissimilar materials under plane extension. However, it is not difficult to show that this oscillation occurs at a distance close to the crack tip. To this end, the value of t at which $(\sigma_y)_{y=0}$ changes sign will be determined from

$$\kappa\log\left(\frac{t+a}{t-a}\right) = \pm\frac{\pi}{2}, \quad |t| > a$$

where κ is a bi-elastic constant given by eq. (29). It follows that

$$a = \pm t\tanh\left[\frac{\pi^2}{2\log\beta}\right], \quad |t| > a \tag{53}$$

in which

$$\beta = \frac{\mu_2}{\mu_1}\left(\frac{\gamma\mu_1+1}{\gamma+\mu_2}\right)$$

For a bi-material plate with elastic properties $E_1 = 3.1\times 10^7$ psi, $E_2 = 10^7$ psi, $\nu_1 = 0.30$, and $\nu_2 = 0.22$, the constant β equals 1.831. If r denotes the radial distance measured from the crack tip, then $t-a = r$ and the first zero of $(\sigma_y)_{y=0}$ takes place when

$$r/a = 1.64\times 10^{-7} \tag{54}$$

It is apparent that the oscillation of stress is confined to very small neighborhoods of the end of the crack where the infinitesimal theory of elasticity is no longer valid.

As remarked in [3, 5] concerning the plane problems of cracks, there is a tendency of the upper surface of the crack to interfere with the lower. Such a phenomenon can also be observed in the present problem of flexure of thin plates. Computing for the normal displacements of the crack surfaces from eqs. (5) and (50), the result is

$$v_1^+ - v_2^- = -\frac{\delta M}{\pi\mu_2 D_1(3+\nu_1)}\left[\frac{\gamma\mu_1(1+\mu_2)+\mu_2(1+\mu_1)}{1+4\varkappa^2}\right]\left[\cos(\varkappa\log\frac{r}{\ell}) + 2\varkappa\sin(\varkappa\log\frac{r}{\ell})\right]\left(\frac{r}{\ell}\right)^{\frac{1}{2}} \tag{55}$$

where $\ell = 2a$, the total length of the crack. Thus, the sign of the normal displacements is seen to change rapidly as the crack tip is approached so that effectively certain parts of the crack boundary may overlap one another. On physical grounds, this kind of interference is not admissible. Moreover, for cracks under flexure, further complication arises due to the fact that crack edges on the compression side can make contact. Such an implication has been ignored in the solution of the present problem and must be taken into account when the theory is verified experimentally.

Periodically Spaced Cracks at the Junction of Two Joined Plates

The solution for the problem of an infinite series of equal cracks of length 2a and spaced at constant intervals b (> 2a) along the interface of two plates bonded together can be obtained in a manner similar to that of a single line crack. The bi-material plate is infinite in extent with uniform bending moments $(M_x^\infty)_1$, $(M_y^\infty)_2$, and M_y^∞ prescribed at infinity.

Starting from eqs. (40), the functions $P_n(z)$ and $X(z)$ must be rearranged for (2n+1) equal segments by letting

$$a_j = jb-a, \quad b_j = jb+a, \quad j = 0, \pm 1, \pm 2, \ldots, \pm n \tag{56}$$

Thus, the Plemelj function may be written as

$$X(z) = (-1)^n b^{2n}(n!)^2\left\{(z+a)^{\frac{1}{2}-i\varkappa}\prod_{j=1}^{n}\left[1-\left(\frac{z+a}{jb}\right)^2\right]^{\frac{1}{2}-i\varkappa} \cdot (z-a)^{\frac{1}{2}+i\varkappa}\prod_{j=1}^{n}\left[1-\left(\frac{z-a}{jb}\right)^2\right]^{\frac{1}{2}+i\varkappa}\right\} \tag{57}$$

and $P_n(z)$ becomes

$$P_n(z) = A_o(-1)^n b^{2n}(n!)^2(z+c)\prod_{j=1}^{n}\left[1-\left(\frac{z+c}{jb}\right)^2\right] \tag{58}$$

where $c = A_1/A_o$. Knowing that

$$(z+d)\prod_{j=1}^{\infty}\left[1-\left(\frac{z+d}{jb}\right)^2\right] = \frac{b}{\pi}\sin\left[\frac{\pi(z+d)}{b}\right]$$

in which d may stand for c or $\pm a$ in eqs. (57) and (58), the ratio $P_n(z)/X(z)$ tends to the limit

$$\frac{P_n(z)}{X(z)} = A_o\sin\left[\frac{\pi(z+c)}{b}\right]\left[\sin^2\left(\frac{\pi z}{b}\right)-\sin^2\left(\frac{\pi a}{b}\right)\right]^{-\frac{1}{2}}\left\{\frac{\sin\left[\frac{\pi(z+a)}{b}\right]}{\sin\left[\frac{\pi(z-a)}{b}\right]}\right\}^{i\varkappa} \tag{59}$$

as $n \to \infty$. The constant c may be determined from eq. (37):

$$\int_{-a}^{a}\left[\frac{\sin\frac{\pi(t+a)}{b}}{\sin\frac{\pi(t-a)}{b}}\right]^{i\varkappa}\frac{\sin\left[\frac{\pi(t+c)}{b}\right]}{\sqrt{\sin^2\left(\frac{\pi t}{b}\right)-\sin^2\left(\frac{\pi a}{b}\right)}}\,dt = 0 \tag{60}$$

Upon integration, c is found to be

$$c = -2i\varkappa a$$

and eq. (38) is also satisfied. The final solution in terms of $F_j(z)$, $j = 3, 4$, is

$$F_3(z) = \frac{1}{1-\gamma}\left[\gamma(\gamma\mu_1+1)\Gamma_1 - (\gamma+\mu_2)\Gamma_2\right]$$

$$F_4(z) = -\frac{(1+\mu_1)M_y^\infty}{4D_1}\sin\left[\frac{\pi(z-2i\varkappa a)}{b}\right]\left[\sin^2\left(\frac{\pi z}{b}\right)-\sin^2\left(\frac{\pi a}{b}\right)\right]^{-\frac{1}{2}}\left[\frac{\sin\frac{\pi(z+a)}{b}}{\sin\frac{\pi(z-a)}{b}}\right]^{i\varkappa} \tag{61}$$

Eqs. (61) provide the correct boundary conditions at infinity. When b gets sufficiently large, eqs. (61) reduce to the special case of a single line crack, eqs. (40).

Fracture Criterion of Cracks in Mixed Media under Flexure

Of fundamental interest is the interpretation of current fracture mechanics theories to cracks in mixed media under flexure. In order to be definite, the concept of stress-intensity factor [10] K_j, $j = 1, 2$, used in the Griffith-Irwin theory of fracture [11], will be introduced. The theory is based upon a detailed analysis of the elastic stress field near a crack tip. Therefore, it is limited to brittle fracture with the possible extension to certain situations where the material in a small region around the crack tip may yield but not appreciably enough to seriously disturb the stress distribution outside of this region.

Now, the necessary information leading to fracture may be obtained from eqs. (23) and (24). Near a crack tip, say, $b_j = z_j$, the functions $\Phi_j(z)$ and $\Omega_j(z)$ may be approximated by

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in which d may stand for c or ±a in eqs. (57) and (58), the ratio $P_n(z)/X(z)$ tends to the limit

$$\frac{P_n(z)}{X(z)} = A_o \sin\left[\frac{\pi(z+c)}{b}\right]\left[\sin^2\left(\frac{\pi z}{b}\right) - \sin^2\left(\frac{\pi a}{b}\right)\right]^{-\frac{1}{2}} \left\{\frac{\sin\left[\frac{\pi(z+a)}{b}\right]}{\sin\left[\frac{\pi(z-a)}{b}\right]}\right\}^{i\varkappa} \tag{59}$$

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Now, the necessary information leading to fracture may be obtained from eqs. (23) and (24). Near a crack tip, say, $b_j = z_j$, the functions $\Phi_j(z)$ and $\Omega_j(z)$ may be approximated by

$$\Phi_1(z) = (\gamma+\mu_2)G_j^{(1)}(z)(z-z_j)^{-\frac{1}{2}-i\kappa}+G_j^{(2)}(z)$$

$$\Phi_2(z) = \gamma(\gamma\mu_1+1)G_j^{(1)}(z)(z-z_j)^{-\frac{1}{2}-i\kappa}+G_j^{(3)}(z)$$

and (62)

$$\Omega_1(z) = -\mu_2(\gamma\mu_1+1)G_j^{(1)}(z)(z-z_j)^{-\frac{1}{2}-i\kappa}+G_j^{(4)}(z)$$

$$\Omega_2(z) = -\gamma\mu_1(\gamma+\mu_2)G_j^{(1)}(z)(z-z_j)^{-\frac{1}{2}-i\kappa}+G_j^{(5)}(z)$$

where

$$G_j^{(k)}(z) = \sum_{n=0}^{\infty} C_{jn}^{(k)}(z-z_j)^n , \quad k = 1, 2, \ldots, 5 \tag{63}$$

are functions that tend to definite limits as $z \to z_j$. While the magnitude of the stress field near z_j can be described by the leading term, say $C_{jo}^{(1)} = C$, in eq. (63), it is convenient to define

$$K = K_1 - iK_2 = -\frac{12\sqrt{2D_1(3+\nu_1)}}{h^2}\left[\frac{\mu_2}{\mu_1}(\gamma+\mu_2)(\gamma\mu_1+1)\right]^{\frac{1}{2}} C \tag{64}$$

so that when $\mu_1 = \mu_2$ and $\gamma = 1$, eq. (64) reduces to the definition of K_j, $j = 1, 2$, used in the homogeneous case [10]. Eqs. (62) to (64) imply that in the limit as $z \to z_j$, the stress-intensity factors K_j may be found from the formula

$$K = -\frac{12\sqrt{2D_1(3+\nu_1)}}{h^2}\left[\frac{\mu_2}{\mu_1}\left(\frac{\gamma\mu_1+1}{\gamma+\mu_2}\right)\right]^{\frac{1}{2}} \lim_{z\to z_j}(z-z_j)^{\frac{1}{2}+i\kappa}\,\Phi_1(z) \tag{65}$$

Once the critical values of K_j, $j = 1, 2$, are determined experimentally for a given combination of the two bonded materials, they may be used to predict the onset of rapid fracture of cracks under the action of flexural loads. It is reminded again that special attention must be given to the measurement of K_j in connection with the contact of crack edges on the compression side of the plate. Nevertheless, it is of interest to list the results of K_j for some basic problems of practical importance. The following values of K_j are computed from eq. (65) together with the stress functions of the various examples discussed before.

(1) Single crack of length 2a is situated along the bond line of two plates of dissimilar materials with uniform bending moments M_y^{∞} at large distances from the crack.

$$K_1 = \frac{6M_y^{\infty}}{h^2}\frac{\sqrt{a}}{\cosh\pi\kappa}[\cos(\kappa\log 2a) + 2\kappa\sin(\kappa\log 2a)]$$

$$K_2 = \frac{6M_y^{\infty}}{h^2}\frac{\sqrt{a}}{\cosh\pi\kappa}[2\kappa\cos(\kappa\log 2a) - \sin(\kappa\log 2a)] \tag{66}$$

(2) Equal and opposite couples of magnitude M are applied to the crack surfaces at $x = b$, where the line crack extends from -a to a (>b) and lies between two joined plates.

$$K_1 = -\frac{6M}{h^2}\frac{1}{\pi\sqrt{a}}\sqrt{\frac{a+b}{a-b}}\cos\left\{\kappa\log\left[2a\left(\frac{a-b}{a+b}\right)\right]\right\}$$

$$K_2 = \frac{6M}{h^2}\frac{1}{\pi\sqrt{a}}\sqrt{\frac{a+b}{a-b}}\sin\left\{\kappa\log\left[2a\left(\frac{a-b}{a+b}\right)\right]\right\} \tag{67}$$

(3) A row of straight cracks each of length 2a are equally spaced with distance b along the interface of a bi-material plate subjected to bending moments M_y^{∞} at infinity.

$$K_1 = \frac{6M_y^{\infty}}{h^2}\left(\frac{b}{\pi}\right)^{\frac{1}{2}}\cosh^{-1}(\pi\kappa)\left(\tan^{\frac{1}{2}}\left(\frac{\pi a}{b}\right)\cosh\left(\frac{2\pi a}{b}\kappa\right)\cos\left\{\kappa\log\left[\frac{b}{\pi}\sin\left(\frac{2\pi a}{b}\right)\right]\right\} + \cot^{\frac{1}{2}}\left(\frac{\pi a}{b}\right)\sinh\left(\frac{2\pi a}{b}\kappa\right)\sin\left\{\kappa\log\left[\frac{b}{\pi}\sin\left(\frac{2\pi a}{b}\right)\right]\right\}\right)$$

$$K_2 = \frac{6M_y^{\infty}}{h^2}\left(\frac{b}{\pi}\right)^{\frac{1}{2}}\cosh^{-1}(\pi\kappa)\left(\cot^{\frac{1}{2}}\left(\frac{\pi a}{b}\right)\sinh\left(\frac{2\pi a}{b}\kappa\right)\cos\left\{\kappa\log\left[\frac{b}{\pi}\sin\left(\frac{2\pi a}{b}\right)\right]\right\} - \tan^{\frac{1}{2}}\left(\frac{\pi a}{b}\right)\cosh\left(\frac{2\pi a}{b}\kappa\right)\sin\left\{\kappa\log\left[\frac{b}{\pi}\sin\left(\frac{2\pi a}{b}\right)\right]\right\}\right) \tag{68}$$

Eqs. (68) reduce to eqs. (66) when b is large in comparison with a. In the homogeneous case, they become

$$K_1 = \frac{6M_y^{\infty}}{h^2}\sqrt{\frac{b}{\pi}\tan\left(\frac{\pi a}{b}\right)} , \qquad K_2 = 0 \tag{69}$$

which is similar to the corresponding problem of plane extension.

The dependency of K_j, $j = 1, 2$, on the bi-material constant κ in eqs. (66) to (68) show that cracks found at the interface of two dissimilar materials will not extend in a planar fashion even if the bending moments were applied symmetrically with respect to the crack line. Unlike the homogeneous problem ($\kappa = 0$) of symmetrical bending, where a single stress-intensity factor is sufficient, the bi-material problem requires both K_1 and K_2 as illustrated by eqs. (69) and (68). Hence, the simple extension of the Griffith-Irwin theory of fracture to cracks in dissimilar media must assume that the combination of K_1 and K_2 will cause the onset of rapid fracture upon reaching some critical value, say f_{cr}, i.e.,

$$f(K_1, K_2) = f_{cr} \tag{70}$$

regardless of the symmetry conditions of the external loads. The form of $f(K_1, K_2)$ can be found experimentally for various combinations of the bonded materials.

Conclusions

The flexural problem of two bonded plates of different materials with cracks along the bond line has been reduced to the problem of Hilbert in complex function theory. The method of solution described earlier can be transferred to the problem of an infinite plate containing a circular insert of another material partially joined along a finite number of arcs.

Oscillation of the stresses and wrinkling of the displacements were observed in regions extremely close to the crack tip, where the linear theory of elasticity fails to hold. Such phenomena are not uncommon and can also be found for cracks in homogeneous materials subjected to mixed boundary conditions. For example, the problem of a straight rigid bar welded to the lower edge of a crack, while the upper edge is free from tractions, can be easily solved to show that the stresses and displacements near the crack tip change sign an infinite number of times.

It should be pointed out that since Kirchhoff boundary conditions were used, the present solutions will not be accurate near the crack boundary. However, the order of the crack-tip stress singularities derived from the Kirchhoff theory is not expected to change were the problem solved by higher order theory such as that developed by Reissner[12]. This is evidenced by the analogous bending problem of cracks in homogeneous plates discussed by Knowles and Wang [13]. From the fracture mechanics point of view, therefore, the stress-intensity factors K_j, defined in the present paper, may be used with sufficient confidence to determine the critical length of cracks in bi-material plates.

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