A-18 AN EXAMINATION OF THE FRACTURE MECHANICS ENERGY BALANCE FROM THE POINT OF VIEW OF CONTINUUM MECHANICS

James R. Rice*

ABSTRACT

The Griffith energy balance for fracture with extensions to inelastic materials considers a cracked body as a linear elastic continuum in which the potential energy released by a crack extension should balance the surface energy plus the energy dissipated by inelastic deformation at the fracture load. With progress in continuum mechanics analyses of crack tip stress fields for material models other than purely linear elastic behavior (non-linear elastic, elastic - plastic, visco - elastic, visco - plastic, etc.) the possibility arises that deviations from linear elastic behavior may form a predictable part of the mechanics rather than an effect treatable only by inclusion of a modified surface energy term. This paper presents an examination and discussion of the fracture mechanics energy balance from this more general viewpoint, attempting to seek those conclusions which follow from theorems and methods of continuum mechanics and broad classifications of continua, rather than from specific and largely unavailable inelastic deformation analyses.

A Griffith type fracture criterion is employed in that it is assumed for crack extension that the work of applied forces must equal the sum of the strain energy change, kinetic energy change, energy dissipated by inelastic deformation, and surface energy. All energy variations except the surface energy are assumed estimated from a continuum solution for an advancing crack satisfying the equations of continuum mechanics and constitutive relations appropriate to the material, while the surface energy is assumed independently known from microstructural considerations. Under this Griffith type assumption it is shown, irrespective of the particular constitutive relation employed, that the fracture criterion is determined solely by local stresses and deformations near the crack tip (or mathematically, by crack tip singularities in continuum solutions), and that an overall Griffith energy balance is equivalent to setting the work done in stress removal from the new crack surface as estimated by the continuum analysis equal to the independent work estimate for bond breakage in the form of surface energy. While all conclusions of the paper tacitly assume the validity of a Griffith type fracture criterion, the inadequacy of such a criterion for prevalent highly ductile fracture mechanisms such as void coalescence by intense plastic flow (rather than fracture by direct bond separation) is emphasized.

---

* Assistant Professor of Engineering, Brown University, Providence, R.I., U.S.A.

Some general results for crack extension in stable linear or non-linear elastic materials are given, and recent proposals pertaining to the influence on the fracture criterion of uniform non-singular stress states, arising in biaxial tension, are shown inappropriate. The influences of surface energy and hardening behavior in determining fracture conditions in elastic-plastic materials are discussed, and important differences of interpretation arise with existing proposals of energy criteria for brittle-like fracture in ductile materials. In particular, a consequence of the Griffith type assumption discussed above is that the surface energy term is of major importance in determining fracture strength, even though its magnitude is commonly negligible in comparison to the plastic dissipation. This is because the difference between potential energy released and plastic energy dissipated in a (hypothetical) crack advance is a function of the applied load level, equal to the surface energy at fracture. The necessity of including hardening behavior in a material description is emphasized for situations in which the Griffith criterion is physically appropriate, for it is proven that with the perfectly plastic idealization this criterion is never satisfied in the sense that the energy surplus required for surface energy cannot be attained. Equivalently, fracture according to the Griffith type assumption can never occur in perfectly plastic materials. The roles of surface energy and hardening behavior in fracture of elastic - plastic materials are further clarified by the analysis of a highly simplified model for crack extension and by dimensional considerations. At low stress levels inducing plastic behavior on a small scale compared to cracked body dimensions, the Griffith assumption leads to a potential energy release at fracture proportional to the surface energy, the coefficient of proportionality depending on plastic stress strain relations and generally increasing with decreasing hardening behavior. While such conclusions are consistent with known environmental influences, it is cautioned that they apply only when the Griffith bond breakage mechanism adequately reflects the actual separation process in ductile materials.

## INTRODUCTION

The explanation of fracture in terms of an energy balance for the extension of pre-existing cracks began with the classic work of Griffith[1], who obtained a criterion of brittle fracture by equating the decrease in potential energy of a linearly elastic body, due to crack extension, to the energy of the newly created surface. The Griffith theory gave a reasonably good agreement with experimental results for materials such as glass and, when combined with a statistical flaw theory, successfully explained the great increase in rupture strength of glass whiskers. Fracture in the technologically important materials is usually accompanied by irreversible plastic and/or viscous deformation near a crack tip, and one is naturally led to





consideration of a modified Griffith-type theory which accounts for this behavior. Such modifications for the case of ductile metals were considered by Irwin[2] and Orowan[3] who equated the decrease in elastic potential energy, due to crack extension, to the sum of the energy of the new surface and the work of plastic dissipation. This resulted in fracture criteria identical to those of the Griffith theory, except that now the sum of surface energy and plastic dissipation terms replaced the surface energy term of the Griffith theory. Irwin[4] further elaborated his modification of the Griffith theory by showing that the decrease in potential energy as calculated from the elastic solutions for cracked bodies (or in his terminology, the energy release rate) could always be expressed in terms of the elastic stress intensity factor, which is the coefficient of a characteristic singular term, depending on the inverse square root of distance from the crack tip, in the elastic stress solutions.

The purpose of the following work is to re-examine the energy balance and subsequent failure criteria of fracture mechanics. A general formulation of the Griffith criteria in a form valid for any continuum is presented, and an examination given of resulting fracture criteria for elastic and ductile materials, with particular attention to the role of surface energy and work hardening behavior in determining conditions of failure for the latter.

## ENERGY BALANCE FOR CRACK EXTENSION

A general energy balance for fracture, modeled on the Griffith theory, is presented here in a form valid for any continuous body sustaining a crack. No particular assumptions as to the form of constitutive equations relating stresses and strains are made in deriving the general results of this and the next section. However, resulting expressions are derived under the usual assumptions of infinitesimal deformations so that geometrical non-linearities are ignored.

Consider a cracked continuum, as shown in figure 1, loaded by forces per unit surface area $\underline{T}$ on the portion of bounding surface $A_T$, forces per unit volume $\underline{F}$ throughout the region V occupied by the body, and imposed displacements $\underline{u}$ on the portion of bounding surface $A_U$. Let $\underset{\sim}{\sigma}$ and $\underset{\sim}{\epsilon}$ denote respectively the tensors of stresses and corresponding strains. Referring all quantities to a set of rectangular cartesian coordinates $(x_1, x_2, x_3)$ and using the subscript notation with repeated indices implying a summation over the values 1,2, and 3, the following equations are assumed satisfied:

1) equations of motion $\frac{\partial \sigma_{ij}}{\partial x_j} + F_i = \rho \ddot{u}_i$ throughout V, where the dots denote time derivatives and $\rho$ is the mass density, 2) traction boundary conditions $\sigma_{ij} n_j = T_i$ on $A_T$, where $\underline{n}$ is the unit normal vector drawn

outward from the region under consideration, 3) strain-displacement relations $\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, and 4) displacement boundary conditions that $u_i$ take on prescribed values on $A_U$. To these must be added a constitutive relation for complete specification of a solution.

Suppose that the crack extends, under constant surface tractions on $A_T$ and surface displacements on $A_U$, from an initial state (a) of figure 1(a), at which fracture is imminent, to a state (b) of figure 1(b) so that the traction free crack surface increases by an amount A'. The crack is not necessarily assumed stationary in state (a) and may be propagating with some non-zero velocity with state (a) then being the configuration of the system at some arbitrary fixed instant. Letting superscripts a or b on any mechanical quantity denote its value in the initial or extended states, respectively, the work of the applied forces in the crack extension is

$\int_{A_T} T_i(u_i^b - u_i^a)\,dA + \int_V F_i(u_i^b - u_i^a)dV$. The sum of the change in stored elastic energy and the energy dissipated in the material is $\int_V \left\{ \int_{(a)}^{(b)} \sigma_{ij} d\epsilon_{ij} \right\} dV$, where $\int_{(a)}^{(b)}$ denotes the integral taken over the transition from (a) to (b) or more precisely $\int_{(a)}^{(b)} f dg = \int_{t^a}^{t^b} f\dot{g}dt$, and the change in kinetic energy is $\frac{1}{2}\int_V \rho(\dot{u}_i^b\dot{u}_i^b - \dot{u}_i^a\dot{u}_i^a)dV$. For fracture, the work of applied forces during crack extension is equated to the change in stored energy, dissipated energy, change in kinetic energy, and energy of the newly created surface. It is assumed that all energy terms, except the surface energy, are adequately estimated by the above expressions as evaluated through a continuum mechanics solution for an extending crack. The energy of the newly created surface is denoted by $\Gamma A'$, where $\Gamma$ is the surface energy. Physical interpretations of $\Gamma$, in terms of the work per unit area required to separate surfaces, are discussed in the next section. Ignoring thermal-mechanical coupling (or limiting the discussion to isothermal or adiabatic conditions with appropriate constitutive equations), the energy balance required in the fracture process becomes

$$\int_{A_T} T_i(u_i^b - u_i^a)dA + \int_V F_i(u_i^b - u_i^a)dV$$

$$= \int_V \left\{ \int_{(a)}^{(b)} \sigma_{ij} d\epsilon_{ij} \right\} dV + \frac{1}{2}\int_V \rho(\dot{u}_i^b\dot{u}_i^b - \dot{u}_i^a\dot{u}_i^a)dV + \Gamma A' \,. \tag{1}$$

The meaning of this energy balance is made clear by transforming the terms of (1) in accord with the four above conditions satisfied by the stress and displacement fields. Since $T_i = \sigma_{ij}^b n_j$ on $A_T$, $u_i^b = u_i^a$ on $A_U$, and $\sigma_{ij}^b n_j = 0$ on the newly created traction free crack surface A',

$$\int_{A_T} T_i(u_i^b - u_i^a)dA = \int_{A_T+A_U+A'} \sigma_{ij}^b n_j (u_i^b - u_i^a)dA \tag{2}$$

Applying Green's theorem to transform the above integral on the bounding surface $A_T+A_U+A'$ into an integral on V, and making use of equations of motion and strain-displacement relations,

$$\int_{A_T} T_i(u_i^b - u_i^a)dA = \int_V \frac{\partial}{\partial x_j}\left[\sigma_{ij}^b(u_i^b - u_i^a)\right] dV$$

$$= \int_V \left[\sigma_{ij}^b(\epsilon_{ij}^b - \epsilon_{ij}^a) + \rho\ddot{u}_i^b(u_i^b - u_i^a) - F_i(u_i^b - u_i^a)\right] dV$$

$$= \int_V \left\{ \int_{(a)}^{(b)} \left[\sigma_{ij}^b d\epsilon_{ij} + \rho\ddot{u}_i^b du_i\right] \right\} dV - \int_V F_i(u_i^b - u_i^a)dV \tag{3}$$

It will also be convenient to write

$$\frac{1}{2}\int_V \rho(\dot{u}_i^b\dot{u}_i^b - \dot{u}_i^a\dot{u}_i^a)dV = \int_V \left\{ \int_{(a)}^{(b)} \rho\,\ddot{u}_i du_i \right\} dV \tag{4}$$

Substituting (3) and (4) into the energy balance for crack extension of (1), the fracture criterion becomes

$$\int_V \left\{ \int_{(a)}^{(b)} \left[ (\sigma_{ij}^b - \sigma_{ij})d\epsilon_{ij} + \rho\,(\ddot{u}_i^b - \ddot{u}_i)du_i \right] \right\} dV = \Gamma A' \,. \tag{5}$$

Evaluation of the integral appearing in (5) for a particular material, loading, and geometrical configuration requires a knowledge of appropriate constitutive relations as well as the solution for the stress and deformation fields in the presence of a growing crack. A specific fracture criterion is obtained by dividing (5) by A' and letting this crack extension approach zero. Ignoring dynamic terms (that is, setting $\rho = 0$) one obtains a condition for the load required to produce static fracture analagous to that obtained by Griffith[1] for a linear elastic material. With dynamic terms included, equation (5) may be reviewed as a condition relating applied loadings to the velocity of fracture as in the work of Craggs[5] and Gilman[6] on elastic materials. Other as yet unexplored possibilities exist, particularly for rate sensitive materials such as viscoelastic solids, where, with or without dynamic terms, equation (5) should serve to relate crack velocity and applied loadings.

Some important conclusions may, however, be deduced directly from (5) without recourse to particular constitutive equations. These deal with the actual local nature of the failure criterion based on the over-all energy balance presented here.

Let $V_o$ be any finite volume which completely surrounds the crack tip region of state (a) and which is chosen so that the newly created crack surface, A', of state (b) is inside $V_o$, the bounding surface of $V_o$ being labeled $A_o$. When A' is infinitesimal as in the limiting process required to derive a specific fracture criterion from (5), it is clear that $V_o$ may be chosen arbitrarily small and still satisfy the required conditions. Equation (5) may be written as

$$\int_{V_o}\left\{\int_{(a)}^{(b)}\left[(\sigma^b_{ij}-\sigma_{ij})d\epsilon_{ij}+\rho(\ddot{u}^b_i-\ddot{u}_i)du_i\right]\right\}dV$$
$$+\int_{V-V_o}\left\{\int_{(a)}^{(b)}\left[(\sigma^b_{ij}-\sigma_{ij})d\epsilon_{ij}+\rho(\ddot{u}^b_i-\ddot{u}_i)du_i\right]\right\}dV=\Gamma A'. \qquad (6)$$

Through use of equations of motion, strain displacement relationships, and Green's theorem, the volume integral over $V-V_o$ may be written as

$$\int_{A_T+A_U+A_o}\left\{\int_{(a)}^{(b)}(T^b_i-T_i)du_i\right\}dA\ ,$$

where $T_i=\sigma_{ij}n_j$ and $\underline{n}$ is the unit normal drawn away from the region $V-V_o$. By boundary conditions the integral over $A_T$ and $A_U$ vanishes so that (6) becomes, after dividing by A',

$$\frac{1}{A'}\int_{V_o}\left\{\int_{(a)}^{(b)}\left[(\sigma^b_{ij}-\sigma_{ij})d\epsilon_{ij}+\rho(\ddot{u}^b_i-\ddot{u}_i)du_i\right]\right\}dV$$
$$+\frac{1}{A'}\int_{A_o}\left\{\int_{(a)}^{(b)}(T^b_i-T_i)du_i\right\}dA=\Gamma \qquad (7)$$

Now consider A' to represent an infinitesimal crack extension and assume the stress and deformation fields to be non-singular and continuous everywhere in V except possibly (and probably) at the crack





tip, and to change continuously with crack size in the transition from (a) to (b). These conditions may reasonably be expected to be fulfilled when crack extension is adequately modeled as continuous in the continuum sense so that sharp wave fronts with accompanying stress discontinuities are not encountered. Since points on $A_o$ are at finite distances from the crack tip, the quantity $T^b_i - T_i = (\sigma^b_{ij}-\sigma_{ij})n_j$ is infinitesimal of order A', and since this is integrated over the also infinitesimal $(u^b_i-u^a_i)$ in (7), the integral over $A_o$ is of second order in A' and disappears as $A'\rightarrow o$. Thus the fracture criterion becomes

$$\lim_{A'\rightarrow o}\ \frac{1}{A'}\int_{V_o}\left\{\int_{(a)}^{(b)}\left[(\sigma^b_{ij}-\sigma_{ij})d\epsilon_{ij}+\rho(\ddot{u}^b_i-\ddot{u}_i)du_i\right]\right\}dV=\Gamma, \qquad (8)$$

where $V_o$ is now any arbitrarily small finite region surrounding the crack tip. Any terms in the integrant of (8) which are non-singular at the crack tip make no contribution since $V_o$ may be taken as small as desired and thus, within the framework of continuum mechanics and assuming the validity of a Griffith type theory, the criterion of fracture is determined solely by singularities in the continuum stress and deformation field at the crack tip.

Physically, this means that the Griffith criterion of fracture, though derived from an overall energy balance, is determined by the local stresses and deformations in the immediate vicinity of the crack tip. For conditions of static fracture (dynamic terms omitted from (8)) this result suggests the validity, under a wide range of conditions, of Irwin's proposal[4] that fracture occurs when the crack tip stress intensity factor, as calculated from linear elasticity, attains a critical value. Consider a cracked body which behaves in a predominantly linear elastic fashion except for small regions near the crack tip where response such as plastic, viscous, or other non-linear behavior is activated by high stress levels. If the crack size and other geometric dimensions are sufficiently large so that at fracture dimensions of the non-elastic zone are small in comparison, one may expect that the linear elastic field is little disturbed and that the stress and deformation field in the non-elastic zone is controlled by the stress intensity factor, which determines the strength of the singularity in the linear elastic solution. This has been analytically verified for elastic-perfectly plastic materials[7,8,9,10], although under plane strain conditions the presence of a uniform stress field acting parallel to the crack line may effect some minor modifications. When these conditions of small scale non-elastic behavior are met the elastic stress intensity factor controls the local stress and deformation, and since the fracture criterion (8) depends only on local conditions, one expects fracture to occur when the stress intensity factor attains a critical value characteristic of the material under consideration, temperature of test, plate thickness in the case of thin sheets, and rate of load application when viscous behavior is activated. If the Griffith-type criterion presented here is physically appropriate, the value of the stress intensity factor at fracture for small scale non-elastic behavior and the fracture criterion for cases of large scale non-elastic behavior is predictable if the complete continuum solution for an extending crack is known. In the absence of such solutions, the stress intensity factor provides a means for correlating data. For ductile materials one may go a step further and base fracture criteria on parameters describing local deformations in elastic-perfectly plastic solutions as in [11,9,10], when such are available.

## PHYSICAL INTERPRETATION OF SURFACE ENERGY

The role of surface energy $\Gamma$ in determining conditions of fracture from the local stress and deformation field, and also conditions under which a Griffith formulation is appropriate, are clarified by transforming (5) to an alternate form involving the work of surface tractions. Using equations of motion, strain-displacement relations, and Green's theorem to effect the transformation of (5) from a volume integral to an integral over the bounding surface $A_T + A_U + A'$ of V in state (b),

$$\int_{A_T+A_U+A'} \left\{ \int_{(a)}^{(b)} (T_i^b - T_i)du_i \right\} dA = \Gamma A' , \tag{9}$$

where $T_i = \sigma_{ij}n_j$ and $\underline{n}$ is the unit normal drawn outward from the region V. Since $T_i$ is constant on $A_T$, $u_i$ is constant on $A_U$, and $T_i^b = o$ on A', the new crack surface being traction free in state (b), this becomes

$$-\int_{A'} \left\{ \int_{(a)}^{(b)} T_i du_i \right\} dA = \Gamma A' . \tag{10}$$

Or, in limit form, to obtain a specific fracture criterion,

$$-\lim_{A' \to o} \frac{1}{A'} \int_{A'} \left\{ \int_{(a)}^{(b)} T_i du_i \right\} dA = \Gamma . \tag{11}$$

These equations are, in general, purely formal as the value of the surface traction $\underline{T}$ on the new surface A' changes discontinuously from a possibly infinite value to zero as the crack advances. As discussed in the next section, the integral may be given a definite meaning for elastic materials. However, present and later purposes are served by the clear symbolic meaning of (10) and (11).





The quantity $\int_{A'} \left\{ \int_{(a)}^{(b)} T_i du_i \right\} dA$ represents the work done by surface tractions on A' as the new crack surface is created, and is, under conditions appropriate for fracture, negative since the surfaces are pulled apart under opposing stresses. The left side of (10) is therefore the work done by the new crack surfaces against forces tending to hold them together. It is reasonable, then, to define the surface energy term, $\Gamma$ , in such a way that $\Gamma A'$ is the same work, but as calculated by microstructural considerations of the separation of material surfaces. Depending on the mechanism of fracture, $\Gamma$ so defined may or may not be identical to the surface energy as commonly defined and used by Griffith[1] for fracture in brittle materials such as glass. On the other hand, for fracture in ductile materials $\Gamma$ will not be equal to the plastic dissipation term of the Irwin-Orowan[2,3] modification, for this term includes also work due to plastic flow at material points away from the crack surface, where the work is here assumed estimated by a continuum description with appropriate constitutive relations.

The two estimates of surface work appearing in (10) are independent, and the Griffith criterion predicts fracture when applied loads are sufficiently great to make the continuum estimate agree with the microstructural estimate. Ordinarily, an energy balance is not an extra condition which may be imposed on a mechanical system, but rather a direct consequence of the mechanics involved. The need for an energy balance (or some other criterion based, for example, on an average stress or strain near the crack tip) arises because of the separate mechanical formulations employed, and would not be required if the convenient formulation via continuum mechanics could be adequately replaced by microstructural considerations on an atomic or larger scale depending on the fracture mechanisms involved.

When the fracture mechanism is the brittle type of normal separation of atomic planes through overcoming cohesive forces, $\Gamma$ is appropriately taken as the true surface energy as estimated from atomic force attraction laws as in[6] or by direct measurement as in [12]. For this choice of $\Gamma$ , comparisons may be made with other fracture criteria contemplating similar mechanisms. The approach of Orowan[3] considers a crack as a narrow ellipse with a tip radius of curvature of atomic dimensions, with fracture occuring when the concentrated stress equals the theoretical strength. Barenblatt[13] assumes fracture to occur when stresses in a small region near the crack tip become too great to balance a given pattern of cohesive forces acting on the crack surface, without causing unbounded stresses at the crack tip. The Griffith criterion, being like those of Orowan and Barenblatt a local criterion, is based on a combination of stresses and deformations near the crack tip in the form of energy and is perhaps more appealing on physical grounds in view of the separate continuum and microstructural approaches employed by all. In the case of linear elastic materials where a complete analysis may be carried out it is not surprising that with appropriate values chosen for the physical parameters involved, the criteria of Griffith, Orowan, and Barenblatt lead to essentially equivalent results.

For the common structural metals, the fracture mechanisms involved no longer necessarily suggest a choice of the surface energy term, $\Gamma$, as the usual surface energy associated with cohesive forces. On one extreme, for situations where a high degree of ductile yielding may occur without serious inhibition by work hardening or stress triaxiallity (as for example in the final stages of necking and separation in a tensile test[14] or thin-foil crack extension test[2] of low hardening metals), observations suggest that crack extension is a result of large plastic flow causing a large void (the crack) to coalesce with smaller voids nearby, the latter being either pre-existing or created by the deformation around inhomogenities. Here the fracture propagation is apparently not due to the presence of high stresses enabling the overcoming of cohesive forces, but rather appears to be a purely kinematical consequence of large deformations enabling an apparent crack extension by the flowing together of voids. One may define a surface energy term $\Gamma$ by considering the work required to coalesce a row of voids with the proper mean spacing, and proceede to obtain a fracture criterion through the energy balance.

However, there are some serious objections to the appropriateness of a Griffith type formulation under such circumstances of highly ductile fracture. First the length scale for the fracture process is on the order of the mean void spacing rather than atomic spacings, and a large degree of coupling between the mechanism of crack extension and deformations at points near the crack tip is anticipated. The Griffith formulation as presented here assumes that such interaction can be neglected, as the independent energy estimates of continuum and microstructural mechanics are employed. This coupling would probably leave estimates of elastic energy changes relatively unaffected, but continuum estimates of the plastic work at points near the crack would be questionable. Another objection arises because the fracture mechanism involved is based on the strain required to coalesce voids rather than on the occurrence of high stresses. The energy balance criterion of failure seems, however, to be closely allied to a maximum stress fracture mechanism in the case of elastic-plastic materials. This is suggested by the agreement mentioned above between energy and maximum stress criteria for linear elastic solids, although the proportionality between stress and strain does not make this line of argument especially appealing. A stronger case for the equivalence of maximum stress and energy criteria is provided by results of subsequent sections on application of the energy balance to elastic-plastic solids. The general result is that the more strain near the crack tip is limited by hardening or stress triaxiallity, the lower the fracture strength. This is precisely the opposite of the result





anticipated in cases where crack extension occurs by the kinematics of void coalescence. A more suitable theoretical framework for treatment of highly ductile fractures is suggested by the work of McClintock[15].

The ductile fractures described above are frequently characterized by overall plastic deformation in a structure and, in terms of applied loads, are of the "high stress" type (although the expression is somewhat misleading as a "high stress" fracture in the presence of a small crack could be a low stress fracture with a longer crack). More serious from the point of view of their unexpectedness are the "low stress" type of brittle fractures which may be induced in the usually ductile structural metals by crack like flaws under conditions when plastic flow is inhibited by a low temperature, an alloy constitution, or a mechanical treatment causing increased work hardening and a raised yield point, or by geometrical constraint causing stress triaxiallity, or when chemical alterations serve to reduce the forces required to separate surfaces. The Griffith energy balance here presented, when combined with an adequate continuum treatment of the elastic-plastic solid, appears more suitable for this brittle type of fracture in ordinarily ductile materials, as the objections raised above to application for highly ductile fracture are no longer appropriate. The fact that materials are embrittled by the factors mentioned suggests that the occurrence of large stresses near the crack tip controls failure, and this is, as will be shown, in accord with an energy approach. Further, although on a microstructural level the fracture propagation remains discontinuous, interaction between the mechanism of crack extension and deformation at points near the crack is expected to be less pronounced as compared to the highly ductile case, and thus the continuum model of a continuously extending crack more appropriate for energy estimates near the crack where plastic flow occurs.

As indicated above, the surface energy term $\Gamma$ appearing in the energy balance is appropriately chosen as the work per unit area required to create new surface. Although presumed estimated from microstructural considerations, $\Gamma$ may not be chosen independently of the continuum model employed and reflects to some degree the inadequacies of a continuum treatment. As an extreme case, in the Irwin-Orowan[2,3] modifications the continuum treatment employs only linear elasticity, and due to the inadequacies of such a model, the surface energy term must include, among other things, the total plastic work done on all the material near the crack tip undergoing plastic deformation. This type of inadequacy does not enter present considerations of brittle fracture in elastic-plastic materials, as it is assumed that plastic deformations near the crack are adequately accounted for by the continuum treatment so that contributions to $\Gamma$ come entirely from microstructural features not represented in the continuum model of an extending crack. The aim, then, of the present formulation, as amplified in subsequent sections, is to separate the influences of macroscopicly observable stress-strain behavior and microscopic features of material separation. For the relatively rare cases of pure cleavage in metals, the mode of failure is normal separation by overcoming cohesive forces and the surface energy term, $\Gamma$ , of the present formulation is correctly interpreted as the usual surface energy associated with a free surface.

More generally, such a direct interpretation of $\Gamma$ is not indicated. Due to the mis-orientations of crystal planes from grain to grain, misorientations of grain boundaries (if they provide an easier fracture path), inclusions, and the like, the surface energy term, $\Gamma$ , must include not only the usual surface energy due to the ultimate de-cohesion, but also the energy dissipated in inhomogeneous plastic sliding occuring prior to separation along non-favorably oriented portions of the fracture surface. However, since the same cohesive forces oppose sliding as oppose normal separation, $\Gamma$ is expected to be of the same order of magnitude as the usual surface energy, and any chemical or structural alteration which affects the usual surface energy is expected to similarly affect the surface energy term, $\Gamma$ , appropriate for brittle-like fractures in ductile materials.

Implications of the energy balance as applied to fracture in elastic and elastic-plastic materials are discussed in the following sections. It will be seen that the surface energy term, $\Gamma$ , discussed above has an equally important role in determining fracture conditions from elastic-plastic continuum solutions as does the usual surface energy term employed by Griffith[1] in connection with an elastic solution.

## STATIC FRACTURE IN ELASTIC MATERIALS

Conditions determining fracture in elastic materials, as provided by the energy balance, have been discussed by Rice and Drucker[16] in connection with some general results on the introduction of voids or slits into a loaded stable elastic material. Results, as specialized to crack extension, are given here. The postulate of stability as introduced by Drucker[17,18] defines a class of time independent materials such that for any stress and deformation states (1) and (2), one has

$$\int_{(1)}^{(2)} \left[\sigma_{ij} - \sigma_{ij}^{(1)}\right] d\epsilon_{ij} \geqslant 0, \tag{12}$$

a generalization of the requirement for uniaxial stress that an increase (decrease) in strain causes a corresponding increase (decrease) in stress. For such materials, one may show that crack extension necessarily involves an energy surplus which is available for conversion to surface energy. In terms of (5), with dynamic terms omitted so that





$$\int_V \left\{\int_{(a)}^{(b)} (\sigma_{ij}^{\ b} - \sigma_{ij}) d\epsilon_{ij}\right\} dV = \Gamma A' , \tag{13}$$

this means that the volume integral is non-negative. Since elastic materials are reversible, the integration path may be reversed so that

$$\int_{(a)}^{(b)} (\sigma_{ij}^{\ b} - \sigma_{ij}) d\epsilon_{ij} = \int_{(b)}^{(a)} (\sigma_{ij} - \sigma_{ij}^{\ b}) d\epsilon_{ij}$$ which, by (12), is

non-negative for stable elastic materials, proving the desired result.

This result may also be deduced directly from the theorem of minimum potential energy (which is implied by material stability). Letting

$$W(\underset{\sim}{\epsilon}) = \int_o^{\epsilon} \sigma_{ij} d\epsilon_{ij}$$ be the strain energy density, the potential

energy

$$P = \int_V W(\underset{\sim}{\epsilon}) dV - \int_{A_T} T_i u_i dA - \int_V F_i u_i dV \tag{14}$$

takes on, in the equilibrium state, an absolute minimum on the class of all compatible displacement fields satisfying displacement boundary conditions on $A_U$. Comparing with (1) and omitting dynamic terms the energy balance becomes

$$P^a - P^b = \Gamma A' \tag{15}$$

with the requirement for an energy surplus being $P^b \leqslant P^a$. The displacement field $u_i^a$ of the initial state (a) clearly is compatible and satisfies displacement boundary conditions for the extended state (b). Thus by the minimum principle, since surface tractions on $A_T$ and body forces remain unchanged, $P^b \leqslant P^a$, proving again that crack extension necessarily creates an energy surplus.

Consider two different elastic bodies, labeled (1) and (2), of the same material with both sustaining cracks, and suppose the loadings are such that in identical arbitrarily small volumes $V_o$ surrounding the crack tips both have the same initial stress state $\sigma_{ij}^{a1} = \sigma_{ij}^{a2} \equiv \sigma_{ij}^{a}$ before crack extension. Equilibrium then also requires $F_i^1 = F_i^2 \equiv F_i$ in $V_o$. It is shown below that if conditions for fracture are met in (1), they are also met in (2). This result is closely related to the local nature of the fracture criterion obtained from the energy balance; as a special case one has the result that for identical bodies the fracture criterion is independent of the method of load application (imposed surface tractions, imposed displacements, or any combination) provided that the same stress state results near the crack tip for all methods of loading, as proven for linear elastic materials by

Irwin[19], Bueckner[20], and Sanders[21].

Dropping dynamic terms from (8), it is clear that both bodies (1) and (2) simultaneously meet fracture conditions if the difference

$$\Delta(A') = \int_{V_o} \left\{ \int_{(a)}^{(b2)} (\sigma_{ij}^{b2} - \sigma_{ij}) d\epsilon_{ij} - \int_{(a)}^{(b1)} (\sigma_{ij}^{b1} - \sigma_{ij}) d\epsilon_{ij} \right\} dV \qquad (16)$$

is of second order in an infinitesimal crack extension A', since then the limit of (8) is identical for both. Here the notations (b1) and (b2) denote the respective states after crack extension A'. Assuming a strain energy function exists so that the integrals are path independent, (16) may be put in the form

$$\Delta(A') = \int_{V_o} \left\{ (\sigma_{ij}^{b2} - \sigma_{ij}^{b1})(\epsilon_{ij}^{b1} - \epsilon_{ij}^{a}) + \int_{(b1)}^{(b2)} (\sigma_{ij}^{b2} - \sigma_{ij}) d\epsilon_{ij} \right\} dV. \qquad (17)$$

Considering only stress paths satisfying $T_i = 0$ on A' in the inner integral, an application of Green's theorem leads to

$$\Delta(A') = \int_{A_o} \left\{ (T_i^{b2} - T_i^{b1})(u_i^{b1} - u_i^{a}) + \int_{(b1)}^{(b2)} (T_i^{b2} - T_i) du_i \right\} dA, \qquad (18)$$

where $A_o$ is the bounding surface of $V_o$, $T_i = \sigma_{ij} n_j$ with $\underline{n}$ the outward normal from $V_o$, and where the usual terms containing body forces cancel. The term $T_i$ represents any set of surface tractions on $A_o$ which pass from those of state (b1) to (b2) while remaining in equilibrium with body forces $F_i$ in $V_o$. Such a set may clearly be chosen with $T_i$ between $T_i^{b1}$ and $T_i^{b2}$ at every point of $A_o$. Since points on $A_o$ are at finite distances from the crack tip, continuity of the solution then requires that all terms in (18), and thus $\Delta(A')$, be of second order in A', so that if the fracture criterion of equation (8) is satisfied for one body, it is also satisfied for the other.

When the material is linearly elastic, the existence of a strain energy function results in $\int_{(a)}^{(b)} (\sigma_{ij}^{b} - \sigma_{ij}) d\epsilon_{ij} = \frac{1}{2} (\sigma_{ij}^{b} - \sigma_{ij}^{a})(\epsilon_{ij}^{b} - \epsilon_{ij}^{a})$. Thus the fracture criterion of (8) and (11) becomes

$$\lim_{A' \to o} \frac{1}{2A'} \int_{V_o} (\sigma_{ij}^{b} - \sigma_{ij}^{a})(\epsilon_{ij}^{b} - \epsilon_{ij}^{a}) dV$$

$$= - \lim_{A' \to o} \frac{1}{2A'} \int_{A'} T_i^{a} u_i^{b} dA = \Gamma . \qquad (19)$$





The latter form of (19) is the equation used by Irwin[4] to express the energy release rate as a quadratic function of elastic stress intensity factors for the opening, sliding, and warping modes of crack extension, thus verifying his result.

Equation (19) may be used to evaluate recent conclusions by Swedlow[22], recorded also by Berry[23], that a uniform stress field acting paralled to a straight crack, and inducing no stress singularity, may influence the results of an energy balance. Considering a crack of length $\ell$ under plane strain conditions subjected to stresses $S_x$ imposed parallel and $S_y$ imposed perpendicular to the crack line, as in figure 2, it is reported[23] that when the plane is infinite an energy balance based on linear elasticity leads to $(\pi \ell / 32G) \times \left[ S_y^2 - (1-4\nu) S_x S_y \right] = \Gamma$, where G and $\nu$ are the shear modulus and Poisson ratio. The inappropriateness of this result is readily shown. The energy balance when transformed to (19) involves only the stress and strain differences $(\sigma_{ij}^{b} - \sigma_{ij}^{a})$ and $(\epsilon_{ij}^{b} - \epsilon_{ij}^{a})$ due to an increment of crack extension. For a linear material, superposition indicates that these differences are the sum of differences induced separately by the loadings $S_x$ and $S_y$. But $S_x$ induces uniform stress and strain fields which are unaffected by crack extension, so that the contribution of $S_x$ to $(\sigma_{ij}^{b} - \sigma_{ij}^{a})$ and $(\epsilon_{ij}^{b} - \epsilon_{ij}^{a})$ is zero, and the result of an energy balance is independent of parallel stress $S_x$, in conflict with [22].

Alternately it was shown earlier that, as a direct result of the minimum potential energy principle, crack extension necessarily makes the potential energy difference $(P^a - P^b)$ non-negative. Comparing (15) in limit form, $\lim_{A' \to o} \frac{1}{A'} (P^a - P^b) = \Gamma$, with the result of[22] as given above, it is seen possible to choose $S_x$ so that the term corresponding to the potential energy difference is made negative. Thus the validity of the cited result of [22] would be a contradiction of the minimum potential energy principle, the latter being a well known direct consequence of the equations of linear elasticity.

## STATIC FRACTURE IN ELASTIC-PLASTIC MATERIALS

Under conditions accompanying brittle fracture in ordinarily ductile elastic-plastic materials, the energy balance, with proper interpretation of the surface energy term and a reasonably descriptive continuum solution, may be expected to provide a criterion for crack extension. General results equivalent to those cited for elastic materials have not been obtained, as the proofs depend on the existence of a strain energy function. However, some interesting results on the role of surface energy, $\Gamma$, and the necessity of work hardening behavior in a continuum description are indicated.

It will be convenient for purposes of clarity to write the energy balance of (1) in terms of the familiar energy release rate as introduced by Irwin[2,4]. Consider a straight, through the thickness crack in an elastic-plastic plane as in figures 3 (a) and (b) under conditions inducing a stress state depending only on $x_1 = x$ and $x_2 = y$, and base all calculations on a unit thickness in the $x_3 = z$ direction. Then $\Delta\ell = A'/2$ of figure 3 is the increment of crack length (A' being the area of both sides of newly created surface). Strain increments may be split into a recoverable elastic part $d\epsilon_{ij}^{e}$ and a permanent plastic part $d\epsilon_{ij}^{p}$, with $d\epsilon_{ij} = d\epsilon_{ij}^{e} + d\epsilon_{ij}^{p}$. The energy release rate, $\mathcal{G}$, as introduced by Irwin is the negative of the rate of change of potential energy, $P = \int_V \left\{ \int \sigma_{ij} d\epsilon_{ij}^{e} \right\} dV - \int_V F_i u_i dV - \int_{A_T} T_i u_i dA$, with respect to crack length, V being the volume of a unit thickness of figure 3. Thus

$$\mathcal{G} = \lim_{\Delta\ell \to o} \frac{1}{\Delta\ell} \left[ \int_{A_T} T_i (u_i^b - u_i^a) dA + \int_V F_i (u_i^b - u_i^a) dV - \int_V \left\{ \int_{(a)}^{(b)} \sigma_{ij} d\epsilon_{ij}^{e} \right\} dV \right]. \tag{20}$$

A plastic dissipation rate, p , is defined as the energy irreversibly dissipated by plastic flow during a unit crack extension, as given by the continuum solution. Thus

$$p = \lim_{\Delta\ell \to o} \frac{1}{\Delta\ell} \int_V \left\{ \int_{(a)}^{(b)} \sigma_{ij} d\epsilon_{ij}^{p} \right\} dV . \tag{21}$$

Comparing (20) and (21) to (1) with dynamic terms omitted and noting that $\Delta\ell = A'/2$, the energy balance takes the familiar form

$$\mathcal{G} = p + 2\Gamma . \tag{22}$$

The interpretation of terms on the right is somewhat different from similar equations [2,3,4]. In these works, $\Gamma$ was interpreted as the usual surface energy and p the total plastic work rate; here $\Gamma$ represents a modified surface energy as indicated earlier and p the plastic work rate estimated from continuum considerations. Since the same transformations of the energy balance leading to the forms of (8) and (11) are valid here, one has

$$\mathcal{G} - p = \lim_{\Delta\ell \to o} \frac{1}{\Delta\ell} \int_V \left\{ \int_{(a)}^{(b)} (\sigma_{ij}^{b} - \sigma_{ij}) d\epsilon_{ij} \right\} dV$$
$$= - \lim_{\Delta\ell \to o} \frac{1}{\Delta\ell} \int_{A'} \left\{ \int_{(a)}^{(b)} T_i du_i \right\} dA \tag{23}$$

It is important to note that the quantities $\mathcal{G}$ and p, as calculated from (20) and (21), are obtained directly from an appropriate solution





for an extending crack and depend only on the applied loads. In particular, the plastic dissipation rate is, prior to insertion in a fracture criterion, not a material constant as seems to be commonly assumed, but rather the result of a calculation that may be carried out quite independently of the question as to whether the material under consideration actually will fracture at the given applied loads. Assuming all applied loads in figure 3 proportional to some loading parameter Q, this dependence of the energy rates on applied loads is indicated by $\mathcal{G} = \mathcal{G}(Q)$ and $p = p(Q)$. The fracture criterion (22) then becomes the implicit equation for the value of the loading parameter at fracture

$$\mathcal{G}(Q_f) - p(Q_f) = 2\Gamma , \tag{24}$$

with $Q = Q_f$ at fracture. This equation is depicted in figure 4, where $\mathcal{G}(Q)$ is shown by the solid line and p(Q) by the dashed line. For a perfectly elastic material $p(Q) = 0$ (no plastic dissipation); for an elastic-perfectly plastic material (no work hardening) it is shown later that if stresses vary continuously during crack extension one has $p(Q) = \mathcal{G}(Q)$ for all Q so that the point of fracture is never attained (that is, all the released potential energy is dissipated plasticly, regardless of the applied load level). The difference $\mathcal{G} - p$ in figure 3 is shown as an increasing function of Q, and this turns out to be the case for the simple model analyzed in the next section. More generally, consider the last form of equation (23) for $\mathcal{G}-p$. As the loading parameter Q is increased one expects both the surface tractions $T_i^a$ initially acting on A' and the displacements $u_i^b$, through which they are relaxed in the process of creating new surface, to also increase, suggesting that the negative of work done in removing these tractions and thus, by (23), $\mathcal{G} - p$ are increasing functions of Q.

The essential point which emerges, as a necessary consequence of the Griffith theory applied in conjunction with continuum estimates of energies involved in $\mathcal{G}$ and p, is that the value of the load at fracture is determined by the surface energy with plasticity acting only to alter the functional dependence of $\mathcal{G} - p$ on applied loads. Dimensional considerations of a later section lead, in the case of small scale plastic behavior, to a quantitative estimate of the manner in which fracture strength depends on surface energy. Such a dependence seems not fully appreciated in previous work[2,3] where, since $2\Gamma$ is generally negligible in comparison to the value of p at fracture, the fracture criterion is written as $\mathcal{G}(Q) = p_f$, with $p_f$ considered as a constant characteristic of the plastic dissipation rate at fracture. Surface energy is negligible in this additive sense, but as indicated by figure 3, the value $p_f$ of the function p(Q), at fracture, is essentially determined by $\Gamma$ . Further, for cases where plastic yielding is on a large scale in comparison to geometric dimensions, the functional dependence of p(Q) on Q may be appreciably altered and consequently the value of $p_f$ is not expected to be a fixed material property.

The influence of chemical attack on the mechanical strength of metals has been indicated in[24,25]; in general both the mechanical stress-deformation properties and surface energy are altered. There are, however, situations in which the time of chemical exposure is sufficiently short so that volume diffusion into the material is negligible and stress-deformation properties are essentially unchanged. This may occur in experiments[24] where specimens are fractured immediately after surface wetting, and the embrittling agent reaches material to be fractured by following the newly created crack surface. When stress-deformation properties are unaltered, the functional dependence of $\mathcal{G}$ - p on Q remains as in the virgin material and presumably only the surface energy, $\Gamma$ , is decreased by the embrittling agent present near the crack tip. The geometrical picture of figure 4 then predicts the observed decrease in fracture strength of roughly the same order of magnitude. If a critical plastic dissipation rate controlled fracture, no decrease in strength would be noted in such situations.

It is now shown that if stresses change continuously during crack extension everywhere in V except at the crack tip, an elastic-perfectly plastic material carrying bounded stresses results in $\mathcal{G}$ = p for all levels of applied load, so that crack extension involves no energy surplus and fracture, according to the Griffith theory, cannot occur. This result, though not obvious in the presence of possible strain singularities at the crack tip, suggests the Griffith criterion of fracture to be associated with failure through the production of high local stresses. Splitting strain increments into elastic (recoverable) and plastic parts, (23) for $\mathcal{G}$ - p becomes

$$\mathcal{G} - p = \lim_{\Delta\ell\to 0}\frac{1}{\Delta\ell}\int_{V_o}\left\{\int_{(a)}^{(b)}(\sigma_{ij}^{b}-\sigma_{ij})d\epsilon_{ij}^{e}+\int_{(a)}^{(b)}(\sigma_{ij}^{b}-\sigma_{ij})d\epsilon_{ij}^{p}\right\}dV, \qquad (25)$$

where, as in deriving (8), continuity everywhere but at the crack tip allows the replacement of V by any arbitrarily small finite volume $V_o$ surrounding the crack tip.

Following the continuum theories of perfect plasticity, a fixed convex yield surface in stress space is assumed such that if plastic flow occurs, $\underset{\sim}{\sigma}$ is on the yield surface and the plastic strain increment, $d\underset{\sim}{\epsilon}^p$, has the direction of an outward drawn normal as in figure 5. Consider the term $(\sigma_{ij}^b - \sigma_{ij})d\epsilon_{ij}^p$ , which represents the scaler product of the nine component vectors $(\underset{\sim}{\sigma}^b - \underset{\sim}{\sigma})$ and $d\underset{\sim}{\epsilon}^p$ in figure 5. If the stress state $\underset{\sim}{\sigma}$ does not cause plastic flow, $d\epsilon^p = 0$ and the term vanishes. If $\underset{\sim}{\sigma}$ does cause plastic flow, it is on the yield surface. But for a perfectly plastic material the stress state $\underset{\sim}{\sigma}^b$ resulting at a material point after crack extension is either inside or on the yield surface. Thus the convexity of the yield surface and normality of the plastic strain increment necessarily implies that the scaler product of $(\underset{\sim}{\sigma}^b - \underset{\sim}{\sigma})$ and $d\underset{\sim}{\epsilon}^p$ is non-positive, so that





$$(\sigma_{ij}^{b} - \sigma_{ij})d\epsilon_{ij}^{p} \leqslant 0 \qquad (26)$$

at every point of the material. Alternately, (26) may be taken as a fundamental definition of a class of materials, as by Drucker[17] in his stability postulate, with convexity and normality resulting. With (26), an upper bound for $\mathcal{G}$ - p results by retaining the first integral of (25), so that

$$\mathcal{G} - p \leqslant \lim_{\Delta\ell\to 0}\frac{1}{\Delta\ell}\int_{V_o}\left\{\int_{(a)}^{(b)}(\sigma_{ij}^{b} - \sigma_{ij})d\epsilon_{ij}^{e}\right\}dV . \qquad (27)$$

When the elastic response of the material is linear, Hooke's law relates $\underset{\sim}{\epsilon}^e$ to $\underset{\sim}{\sigma}$ and, if the assumed form admits a strain energy function

$$\int_{(a)}^{(b)}(\sigma_{ij}^{b} - \sigma_{ij})d\epsilon_{ij}^{e} = \tfrac{1}{2}(\sigma_{ij}^{b} - \sigma_{ij}^{a})(\epsilon_{ij}^{eb} - \epsilon_{ij}^{ea})$$ where $\underset{\sim}{\epsilon}^{eb}$ and $\underset{\sim}{\epsilon}^{ea}$ are the elastic strains induced by the stresses $\underset{\sim}{\sigma}^b$ and $\underset{\sim}{\sigma}^a$ . The upper bound of (27) for $\mathcal{G}$ - p becomes

$$\mathcal{G} - p \leqslant \tfrac{1}{2}\lim_{\Delta\ell\to 0}\frac{1}{\Delta\ell}\int_{V_o}(\sigma_{ij}^{b} - \sigma_{ij}^{a})(\epsilon_{ij}^{eb} - \epsilon_{ij}^{ea})dV . \qquad (28)$$

It is now shown that when stresses are bounded everywhere in V and change continuously with crack length everywhere except possibly at the crack tip, the upper bound of (28) is zero.

Take $V_o$ to be the square of side length s and of unit thickness in the z direction, centered at the crack tip of the initial state (a) as in figure 6. The integral over $V_o$ of (28) is

$$\int_{-s/2}^{+s/2}\left\{\int_{\ell-s/2}^{\ell+s/2}(\sigma_{ij}^{b} - \sigma_{ij}^{a})(\epsilon_{ij}^{eb} - \epsilon_{ij}^{ea})dx\right\}dy .$$

Considering the inner integral in x first, curves in figure 5 show the variation of a stress component $\sigma_{ij}$ with x for y = 0 (the crack line), where a bounded discontinuity at the crack tip may occur, and for some non-zero value of y. Since corresponding elastic strain components, $\epsilon_{ij}^{e}$ , are related to stresses by Hooke's law, they have similar variations with x . Continuity then requires that both the stress differences $(\sigma_{ij}^{b} - \sigma_{ij}^{a})$ and elastic strain differences $(\epsilon_{ij}^{eb} - \epsilon_{ij}^{ea})$ be of order $\Delta\ell$ (written as $O[\Delta\ell]$ ) for $y \neq 0$ and for y = 0 except in the discontinuity region $\ell \leqslant x \leqslant \ell + \tfrac{1}{2}\Delta\ell$ . Thus for $y \neq 0$ the bracketed integral in x above is $O[(\Delta\ell)^2]$ . For y = 0 the bounded discontinuity gives $O[\Delta\ell]$ for the integral in x, and integrating over the height s in y results in $\int_{V_o}(\sigma_{ij}^{b} - \sigma_{ij}^{a})(\epsilon_{ij}^{eb} - \epsilon_{ij}^{ea})dV \leqslant s\, O[\Delta\ell]$ . The inequality of (28) then results in $\mathcal{G} - p \leqslant \frac{1}{2} \lim_{\Delta\ell \to 0} \frac{1}{\Delta\ell} \left\{ s\,O[\Delta\ell] \right\} = s\,O[1]$ . But since $V_o$, and therefore s, may be chosen arbitrarily small, this becomes $\mathcal{G} - p \leqslant 0$. Since $\mathcal{G} - p$ is non-negative, one has $\mathcal{G} - p = 0$, or

$$\mathcal{G} \equiv p \tag{29}$$

under the stated assumptions for a perfectly plastic material at all load levels, and the necessary surplus required for surface energy in the fracture criterion $\mathcal{G} = p + 2\Gamma$ is never attained.

This result depends strongly on the assumption that stresses change continuously during crack extension; the assumption may be violated by a finite line sustaining a stress discontinuity and propagating through the material as the crack extends. Available solutions for stationary cracks[7,10,26] indicate the absence of discontinuities, as does the published solution[27,8] for an extending crack under longitudinal shear. One may, in fact, verify through detailed calculation that for the latter solution one has $\mathcal{G} = p$ at all load levels. However, this solution, while perhaps adequate for the purposes of[8], is not exact since by using the stress field appropriate for a stationary crack in the case of a moving crack, the plastic work turns out to be negative at some points of the cracked body. The presence or absence of propagating discontinuities, and thus the general validity of the result $\mathcal{G} = p$, is thus not completely settled.

Assuming the generality of (29), it appears that if a continuum theory is to predict fracture through the Griffith criterion it is necessary to have a stress singularity, or infinite recoverable elastic energy density, at the crack tip for conversion to surface energy, in agreement with the notion that fracture through the Griffith mechanism is equivalent to the building up of sufficiently high crack tip stresses to overcome cohesive forces. This does not indicate that non-work hardening perfect plasticity solutions are without usefulness in situations of brittle fracture. While apparently not being capable of producing an absolute prediction of fracture strength, they do presumably lead to reasonably accurate estimates of how local crack tip deformations depend on applied loadings and geometrical parameters for actual lightly hardening materials.

Aside from indicating the predictive inadequacy of the perfect plasticity model, the above result confirms that as one passes through varying degrees of hardening behavior from the perfect elasticity to the perfect plasticity limiting cases, the difference $\mathcal{G} - p$ passes from $\mathcal{G}$ to zero, as indicated in connection with figure 4. Thus for materials of moderate hardening one is justified in expecting, on the basis of continuum estimates of plastic work, that the observed large values of the energy release rate at fracture, in comparison to results of an elastic analysis, are predictable.

It is of some interest to note that the coincidence of a maximum stress criterion and an energy criterion is more a consequence of the





elastic-plastic model than a general result of the energy balance. Consider a material which is capable of carrying only finite stresses irregardless of the accompanying strains. If the material is perfectly plastic, no energy surplus for fracture is available. On the other hand, if the material is non-linear elastic, a result of the previous section is that crack extension is accompanied by a non-positive change in potential energy. One may easily justify that if stresses change at all during the crack advance (as they must to satisfy boundary conditions if non-zero stresses are transmitted across the new surface before separation), the potential energy change is non-zero and an energy surplus is available for fracture. Thus the common feature for different types of materials appears to be more a requirement of an infinite recoverable energy density at the crack tip than an infinite stress, although for elastic-plastic materials these coincide.

## ENERGY RATE ANALYSIS OF A SIMPLIFIED MODEL

In the absence of appropriate solutions for an extending crack in a work-hardening elastic-plastic material, it is presently not possible to predict fracture strengths from the energy balance( in cases where it is applicable). The situation is to some extent clarified by the analysis of the simplified elastic-plastic model for an extending crack presented here. The model is in no sense quantitatively predictive of the behavior of real materials, and many of the important features of plastic deformation are absent. However, the model does serve to reflect, in a concrete example, the general results presented earlier, and is physically suggestive of the role of surface energy and work-hardening behavior in determining fracture conditions for ductile materials.

Consider two elastic planes subjected to in-plane loadings and joined together along a strip of elastic-plastic material of height h as in figure 7. For simplicity, the strip material is assumed one-dimensional in that it resists extension or contraction only in the y direction (parallel to the direction of applied loadings). No resistance is offered to extension or contraction in the x and z directions so that $\sigma_x = \sigma_z = 0$ in the strip; similarly, no resistance to shear deformation is offered so that all shear stresses vanish in the strip. Figure 8(a) shows a crack of length $\ell$ in the strip and 8(b) the same crack after extension by an amount $\Delta\ell$ , states (a) and (b) corresponding to the initial and extended crack states, respectively, of previous sections. Through the properties of the strip, the cracked material is stress free and strip displacements are discontinuous at the crack tip.

A typical stress-strain relation for the strip material is shown by the curves labeled "loading" and "unloading" in figure 9. The static fracture criterion is $\mathcal{G} = p + 2\Gamma$ ; however, it is not necessary to separately assess the elastic and plastic deformations during crack extension and to write explicit formulae for $\mathcal{G}$ and p, as the fracture criterion requires only that their difference $\mathcal{G}$ - p be known. From (23), as specialized to the present case,

$$\mathcal{G} - p = -\lim_{\Delta\ell\to o}\frac{1}{\Delta\ell}\int_{A'}\left\{\int_{(a)}^{(b)} T_i du_i\right\} dA$$

$$= \lim_{\Delta\ell\to o}\frac{1}{\Delta\ell}\int_{x=\ell}^{x=\ell+\Delta\ell}\left\{\int_{(a)}^{(b)} \sigma_y\, d\left[u_y^{(top)} - u_y^{(bottom)}\right]\right\} dx$$

$$= \left\{\int_{(a)}^{(b)} \sigma_y\, d\left[u_y^{(top)} - u_y^{(bottom)}\right]\right\}_{\text{crack tip}}\,, \qquad (29)'$$

where $\underline{T}$ = (0, $-\sigma_y$, 0) on the top side of the new crack surface, $\underline{T}$ = (0, $\sigma_y$, 0) on bottom side, and the displacement difference arises since the integral on A' is to be carried out on both sides of the newly created surface. The final form of (29)' is to be taken at the crack tip. In state (a), $[u_y^{(top)} - u_y^{(bottom)}] = 0$. Since displacements are continuous in the elastic planes, the only non-infinitesimal displacement differences arise from contraction of the strip material after fracture and, where $\epsilon_y$ is the strain in the strip material at the crack tip during the transition from (a) to (b), $[u_y^{(top)} - u_y^{(bottom)}] = h(\epsilon_y^a - \epsilon_y)$. Thus the difference between the energy release and plastic dissipation rates is, for a material which unloads in a linear elastic fashion as in figure 9,

$$\mathcal{G} - p = -h\int_{(a)}^{(b)} \sigma_y\, d\epsilon_y = \tfrac{1}{2}\, h\sigma_y^a\,(\epsilon_y^a - \epsilon_y^b)\,, \qquad (30)$$

where $\sigma_y^a$ is the stress at the crack tip before extension, and $\epsilon_y^a$ and $\epsilon_y^b$ represent respectively the strains in the strip material before and after extension. Equation (30) may also be derived directly from the alternate form of (23) in terms of the volume integral. Interpreting geometrically through the stress-deformation curves of figure 9, $\mathcal{G}$ - p = h x[shaded area of figure 9] = $h\,\tfrac{1}{2}\sigma_y^a(\epsilon_y^a - \epsilon_y^b)$, which is h times the energy per unit volume recovered by an unloading from the stress level $\sigma_y^a$.

Comparing the fracture criterion $\mathcal{G}$ = p + 2$\Gamma$ with (30), the Griffith formulation predicts fracture in the present case when

$$\tfrac{1}{2}h\sigma_y^a\,(\epsilon_y^a - \epsilon_y^b) = 2\Gamma\,, \qquad (31)$$

or, in terms of figure 9, when the stress $\sigma_y^a$ at the crack tip is sufficiently high to make the shaded area equal $2\Gamma/h$ or, alternately stated, when the stress $\sigma_y^a$ is great enough to make the recoverable





elastic energy balance the surface energy. One may estimate the plastic dissipation rate for the strip model. Supposing the region of the strip undergoing plastic deformation extends from x = $\ell$ to x = $\ell + \omega$ in figure 8, $\omega$ being the length of the plastic zone,

$$p = h\int_{\ell}^{\ell+\omega} \sigma_y \frac{\partial \epsilon_y^p}{\partial \ell}\, dx \qquad (32)$$

is the plastic dissipation rate, with $\sigma_y$ and $\epsilon_y^p$ being the stress and plastic strain in the strip. Consider now cases for which $\epsilon_y^p$ is a function only of x $-\ell$ (the distance from the crack tip) and is otherwise independent of $\ell$. This is usually a good spproximation to the actual case as the variation with crack length of plastic strain at a point some fixed distance ahead of the crack is generally negligible in comparison to the variation with distance from the crack tip. In such cases $\frac{\partial\epsilon_y^p}{\partial x} = -\frac{\partial\epsilon_y^p}{\partial\ell}$ and (32) results in

$$p = h\int_{\ell+\omega}^{\ell} \sigma_y \frac{\partial\epsilon_y^p}{\partial x}\, dx = h\int_{x=\ell+\omega}^{x=\ell} \sigma_y\, d\epsilon_y^p$$

$$= h\int_{o}^{(a)} \sigma_y\, d\epsilon_y^p\,, \qquad (33)$$

since in going from the outer edge ($x = \ell + \omega$) of the plastic zone to the crack tip ($x = \ell$), the plastic strain varies from zero to that of state (a). The integral $\int_o^{(a)}\sigma_y\, d\epsilon_y^p$ is the total irreversible plastic work per unit volume done on the strip material in bringing it to the stress $\sigma_y^a$ at the crack tip, and is simply the unshaded area between the "loading" and "unloading" curves of figure 9. Thus, geometrically, p = h x[unshaded area of figure 9]. The ratio of $\mathcal{G}$ - p to p is then the ratio of the shaded area to the unshaded area of figure 9, so that in cases where appreciable plastic deformation is required to build up sufficient recoverable elastic energy to meet the fracture criterion $\mathcal{G}$ - p = 2$\Gamma$, the value of the plastic dissipation rate, p, at fracture is expected to considerably exceed the surface energy, 2$\Gamma$, in accord with observed results for metals. However, the surface energy determines the value of applied loads at fracture, with plasticity properties serving to determine the functional relation between applied load and recoverable elastic energy.

A qualitative explanation for the influence of work-hardening on fracture strength results by comparing, as in figure 10, failure criteria for strip materials with the same elastic constants but different degrees of hardening. The same crack tip stress, $\sigma_y^a$, is seen to be required for fracture in all cases, but the required strain,

$\epsilon_y^a$, increases with decreasing work-hardening. When the plastic region is sufficiently confined by elastic material surrounding the crack tip, elastic constraint causes the applied load at fracture to increase with crack tip strain required at fracture and, with all other material properties held constant, the applied load at fracture is seen to increase with decreasing work-hardening. Alternately, in terms of energy rates, the potential energy release rate, $\mathcal{G}$, is expected to be essentially unaltered in its dependence on applied load when the yielded region is small and confined by elastic material. But the plastic dissipation rate corresponding to the required recoverable energy surplus for surface energy, and thus the applied load at fracture, increases with decreasing work-hardening.

As pictured in figure 10, when the strip is elastic-perfectly plastic it cannot fracture according to the Griffith criterion, unless fracture conditions are met for a crack tip stress at or below the yield point. Here one does not have $\mathcal{G} - p \equiv 0$, but rather that $\mathcal{G} - p$ is no greater than the recoverable elastic energy corresponding to the yield stress. This is not contradictory with the result of the last section since the assumption of stress continuity during crack extension is not satisfied due to the discontinuity over the finite strip height h. When the strip material is rigid-plastic the model corresponds to that studied in[28,29,9]. Since no energy is recoverable, one has $\mathcal{G} = p$ at all load levels, in disagreement with the analysis of[29] where it was erroneously assumed that the potential energy release rate was unaffected by plastic deformation. It is of some interest to note that when the strip material is assumed to be linear elastic, a fracture criterion identical to that of Griffith[1], except for an undetermined numerical factor, results when the strip height, h, is negligible in comparison to geometrical dimensions of the cracked body. This may be shown by a mathematical formulation, not reproduced here, of the appropriate boundary value problem for the elastic strip analog of the infinite plane with a crack of length $\ell$ under a uniform tensile stress, $\sigma$. The problem may be reduced to a linear integral equation for the stress $\sigma_y$ in the strip. By letting $\ell/h \to \infty$ in such a way that $\sqrt{h/\ell}\,\sigma_y/\sigma$ is bounded and basing the length scale on h, the linearity assures that crack tip stress varies as $\sigma_y^a = (\text{const.})\,\sigma\sqrt{\ell/h}$. Elasticity of the strip requires that $\epsilon_y^b = 0$ after crack extension, and with $\epsilon_y^a = \sigma_y^a/E$ as appropriate for plane stress, the fracture criterion (31) becomes

$$\tfrac{1}{2}\, h\sigma_y^a\,(\epsilon_y^a - \epsilon_y^b) = (\text{const.})\,\frac{\sigma^2\ell}{E} = 2\Gamma, \tag{34}$$

in essential agreement with Griffith except for the constant which may be determined only by a complete solution. Thus the strip model leads to correct results in the two limiting material idealizations of perfect elasticity and perfect plasticity. In general, however, results should be expected strongly dependent on the artificially





introduced strip height so that predictions for work-hardening materials are at best qualitative. The technique described above for the elastic strip is equivalent to obtaining a solution which asymptotically approaches the form of the elastic crack solution as dominated by the crack tip singularity; a similar technique has been shown valid for small scale yielding of a plastic material[10].

## DIMENSIONAL ANALYSIS

The mode of dependence of fracture strength on surface energy for a work-hardening elastic-plastic material may be predicted through dimensional considerations in cases where the yielded zone is small in comparison to geometric dimensions at the point of fracture. Consider an infinite plane sustaining a crack of length $\ell$ and loaded with a uniform tensile stress, $\sigma$, so as to induce plane strain conditions. For an isotropic material, elastic strain increments $d\underset{\sim}{\epsilon}^e$ are related to stress increments $d\underset{\sim}{\sigma}$ by the usual form of Hooke's law involving Young's modulus E and Poisson's ratio $\nu$. An appropriate form[30] for plastic strain increments $d\underset{\sim}{\epsilon}^p$ under the assumption of isotropic hardening of a stable[17,18] material is given in terms of a loading function $f = f(\underset{\sim}{\sigma})$ of the three stress invariants, with $f(\underset{\sim}{\sigma}) = \tau_o$ ($\tau_o$ = initial yield stress) the initial yield surface in stress space, and subsequent yield surfaces determined by the largest value of $f(\underset{\sim}{\sigma})$ attained during the course of previous plastic straining. Components of plastic strain increment are non-zero only for stresses on the current yield surface with stress increments that increase f, in which case

$$d\epsilon_{ij}^p = \Lambda(f)\,\frac{\partial f}{\partial \sigma_{ij}}\,df\,, \tag{35}$$

with $\Lambda(f)$ being the isotropic hardening function. Since $d\underset{\sim}{\epsilon}^p$ is dimensionless and f is a function of stresses, it is clear that any material constants appearing in $\Lambda$ and f may always be expressed in units of stress, so that (35) introduces a set of constants $\tau_1$, $\tau_2$, .., $\tau_n$ with stress dimensions.

The expression for $\mathcal{G} - p$ of (23) as obtained from a continuum solution may at most depend on $\sigma, \ell, E, \nu, \tau_o, \tau_1, \ldots, \tau_n$, with the fracture criterion (22) introducing the additional variable $\Gamma$. The fracture criterion is, however, local in nature. It is well known that for elastic materials, local stresses and deformations depend on applied loads and geometry only through the stress intensity factor K, with $K = \sigma\sqrt{\pi\ell/2}$ in the present case[4]. For perfectly plastic materials, it has been shown in[7,8,9,10] that when the yielded zone is small in comparison to geometric dimensions (or alternately, when $\sigma$ is small in comparison to the limit load), local stresses and deformations remain determined by the stress intensity factor of an elastic solution. Thus one may generally expect in cases of small scale yielding that the local conditions determining $\mathcal{G} - p$ depend on $\sigma$ and $\ell$ only through the combination $K = \sigma\sqrt{\pi\ell/2}$ . For such cases the independent dimensionless combinations of relevant variables are $K/\sqrt{E\Gamma} = \sigma\sqrt{\pi\ell/2E\Gamma}$, $\tau_o/E$, $\tau_1/E$, ..., $\tau_n/E$, $\nu$, so that a plane strain fracture criterion (assuming $\mathcal{G} - p \neq 0$) takes the form

$$K = \sigma\sqrt{\frac{\pi\ell}{2}} = \sqrt{E\Gamma}\, g(\frac{\tau_o}{E}, \frac{\tau_1}{E}, \ldots, \frac{\tau_n}{E}, \nu). \qquad (35)'$$

For comparison, the criterion obtained for elastic materials by Griffith[1] and generalized by Irwin[2] is

$$K = \sigma\sqrt{\frac{\pi\ell}{2}} = \sqrt{E\Gamma}\sqrt{\frac{2}{1-\nu^2}} \quad . \qquad (36)$$

The mode of dependence of fracture strength on surface energy for small scale yielding is then identical to the dependence indicated for elastic materials. In such cases it is reasonable to estimate the potential energy release rate by the linear elastic formula[4] $\mathcal{G} = (1-\nu^2)K^2/E$, so that (35)' leads to a fracture criterion in the form

$$\mathcal{G} = 2\Gamma f(\tau_o/E, \tau_1/E, \ldots, \tau_n/E, \quad ), \qquad (37)$$

indicating that at fracture the energy release rate is directly proportional to the surface energy term as multiplied by some function f depending on material constants describing the elastic-plastic behavior. For linear elastic materials f is unity; previous considerations suggest that as one considers various types of behavior from high to light work-hardening, f increases greatly from unity, approaching infinity in the perfectly plastic case.

It has recently come to the author's attention that expressions indicating a similar dependence on surface energy have been proposed by Gilman[31] and Westwood and Kamdar[32], based on considerations of plastically relaxed stress fields near cracks. The latter work leads to estimates of environmental embrittlement based on alterations of $\Gamma$ due to chemical attack.

The nature of plastic deformation suggests that the functional forms appearing in (35)' and (37) are not unique, but rather dependent on the way stresses acting before crack extension were reached. Thus the criterion for the first increment of crack extension after loading from a virgin state may differ from the criterion for a subsequent increment, since the method of plastic straining involves prior crack extension as well as changes in applied load. Such considerations, although based on a strain criterion instead of an energy balance, have led[11] to an explanation of slow crack growth prior to catastrophic fracture.

In general, dimensional considerations require the introduction of a material property with dimensions of length to obtain a fracture criterion, as crack length is the only parameter entering a continuum





analysis without stress dimensions. This is provided by the ratio $\Gamma/E$ in the Griffith type theory discussed here. For situations involving a kinematical mechanism of void coalescence by plastic flow, a mean void diameter might provide the required length dimension[15]. It appears[33] that dimensional analyses of crack extension rates under repeated loadings (fatigue) may be brought into accord with experimental data by including among relevant variables such a characteristic length associated with fracture.

Williams[34] discusses the application of a Griffith-type energy balance to fracture in visco-elastic materials in another paper presented at this conference. It is of interest to note that, similar to present conclusions for elastic - plastic materials, he finds surface energy to essentially determine fracture strength, even though it is normally small compared to viscous dissipation, when the appropriateness of a Griffith approach is assumed.

## ACKNOWLEDGEMENT

Work reported here was carried out while the author was financially supported by a National Academy of Sciences - National Research Council postdoctoral fellowship funded by the Air Force Office of Scientific Research. This financial assistance, as well as assistance by the Advanced Research Projects Agency in the preparation of the manuscript, is gratefully acknowledged.

Gratitude is expressed to Professor D. C. Drucker and Dr. J. Dvorak, for helpful discussions of this work.

## REFERENCES

1) A. A. Griffith, "The Phenomena of Flow and Rupture in Solids", Phil. Trans. Roy. Soc. London, A221, 1921.

2) G. R. Irwin, "Fracture", in Handbuch der Physik, Springer, Berlin, vol. 6, 1958.

3) E. Orowan, "Energy Criteria of Fracture", Weld. J. Res. Suppl., March, 1955.

4) G. R. Irwin, "Fracture Mechanics", in Structural Mechanics Pergamon Press, 1960.

5) J. W. Craggs, "On the Propagation of a Crack in an Elastic Brittle Material", J. Mech. Phys. Solids, vol. 8, 1960.

6) J. J. Gilman, "Cleavage and Ductility in Crystals", Proc. Int. Conf. on Atomic Mechanisms of Fracture, Wiley, 1959.

7) J. A. H. Hult and F. A. McClintock, "Elastic-Plastic Stress and Strain Distribution Around Sharp Notches Under Repeated Shear", Ninth Int. Cong. Appl. Mech., vol. 8, Brussels, 1956.

8) F. A. McClintock, "On the Plasticity of the Growth of Fatigue Cracks", in Fracture of Solids (eds. Drucker and Gilman), Wiley, 1963.

9) J. R. Rice, "Plastic Yielding Near a Crack Tip", in proc. of this conference.

10) J. R. Rice, "Contained Plastic Deformation Near Cracks and Notches Under Longitudinal Shear", to appear in Int'l. J. Fract. Mech., Sept. 1966 (tentative date).

11) F. A. McClintock and G. R. Irwin, "Plasticity Aspects of Fracture Mechanics", A.S.T.M. Symp. on Crack Toughness Testing and Applications, STP-381, 1965.

12) W. D. Kingery, Property Measurements at High Temperatures, Wiley, New York, 1959.

13) G. I. Barenblatt, "Mathematical Theory of Equilibrium Cracks in Brittle Fracture", in Advances in Applied Mechanics, VII, Academic Press, 1962.

14) H. C. Rogers, "The Tensile Fracture of Ductile Metals", A.I.M.E. Trans., vol. 218, 1960.

15) F. A. McClintock, "A Criterion for Ductile Fracture by the Growth of Holes", Mass. Inst. Tech. Technical Report, 1964.

16) J. R. Rice and D. C. Drucker, "Energy Changes in Stressed Bodies due to Void and Crack Growth", Brown Univ. Report ARPA SD 86/E24, Aug. 1965.

17) D. C. Drucker, "A More Fundamental Approach to Stress Strain Relations", Proc. First Nat. Cong. for Appl. Mech., June 1951, A.S.M.E.

18) D. C. Drucker, "On the Postulate of Stability of Material in the Mechanics on Continua", Jour. de. Mechanique, vol. 3, Juin 1964.

19) G. R. Irwin, "A Critical Energy Rate Analysis of Fracture Strength", Weld. J. Res. Suppl., April 1954.

20) H. F. Bueckner, "The Propagation of Cracks and Energy of Elastic Deformation", Trans. A.S.M.E., August 1958.

21) J. L. Sanders, "On the Griffith-Irwin Fracture Theory", J. Appl. Mech., vol. 27, 1960.

22) J. L. Swedlow, "On Griffith's Theory of Fracture", Galcit SM 63-8, Cal. Inst. Tech., March 1963.

23) J. P. Berry, "Brittle-Like Fracture", in Fracture Processes in Polymeric Solids (ed. B. Rosen), Wiley, 1964.





24) W. Rostoker, J. M. McCaughey, and H. Marcus, Embrittlement by Liquid Metals, Reinhold, New York, 1960.

25) V. I. Likhtman, P. A. Rebinder, and G. V. Karpenko, Effect of a Surface Active Medium on the Deformation of Metals, Her Majesty's Stationary Office, London, 1958 (translation from 1954 original in Russian).

26) L. D. Stimpson and D. M. Eaton, "The Extent of Elasto-Plastic Yielding at the Crack Point of an Externally Notched Plane Stress Tensile Specimen", A.R.L.24, Office of Aerospace Research, Wright-Patterson A.F.B., July 1961.

27) J. A. H. Hult, "Fatigue Crack Propagation in Torsion", J. Mech. Phys. Solids, vol. 6, 1957.

28) D. S. Dugdale, "Yielding of Steel Sheets Containing Slits", J. Mech. Phys. Solids, vol. 8, 1960.

29) J. N. Goodier and F. A. Field, "Plastic Energy Dissipation in Crack Propagation", in Fracture of Solids, Wiley, 1963.

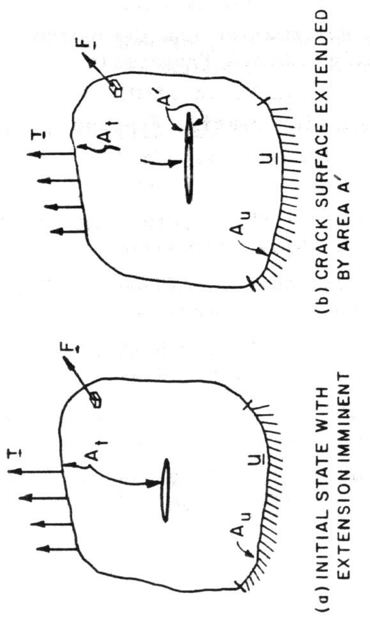
30) D. C. Drucker, "Plasticity", in Structural Mechanics, Pergamon Press, 1960.

31) J. J. Gilman, "Physical Nature of Plastic Flow and Fracture", in Plasticity, Pergamon Press, 1960.

32) A. R. C. Westwood and M. H. Kamdar, "Concerning Liquid Metal Embrittlement, particularly of Zinc Monocrystals by Mercury", Phil. Mag., vol. 8, no. 89, 1963.

33) J. R. Rice, "The Mechanics of Crack Tip Deformation and Extension by Fatigue", A.S.T.M. Symp. on Fatigue Crack Propagation, June 1966.

34) M. L. Williams, "Initiation and Growth of Viscoelastic Fracture", in proc. of this conference.




(a) INITIAL STATE WITH EXTENSION IMMINENT

(b) CRACK SURFACE EXTENDED BY AREA A'

FIG. 1 EXTENDING CRACK

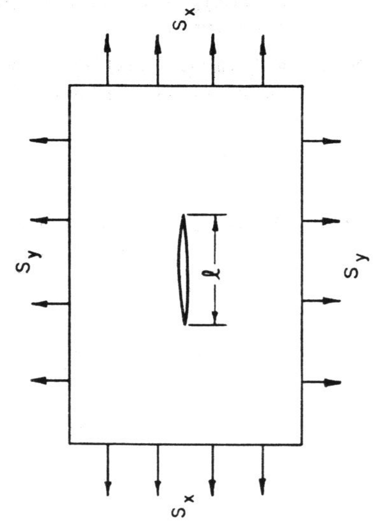
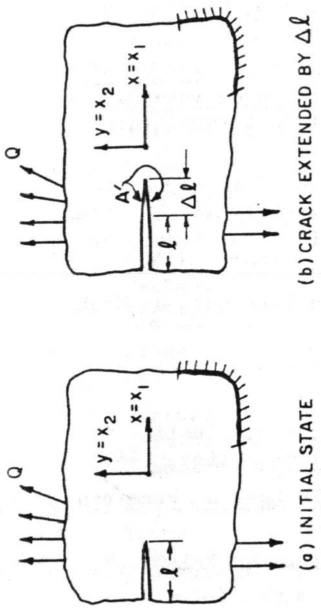



FIG. 2 CRACKED PLANE UNDER BIAXIAL STRESSES




(a) INITIAL STATE

(b) CRACK EXTENDED BY $\Delta \ell$

FIG. 3 CRACK EXTENSION UNDER PLANE CONDITIONS APPROPRIATE FOR DEFINITION OF G AND p

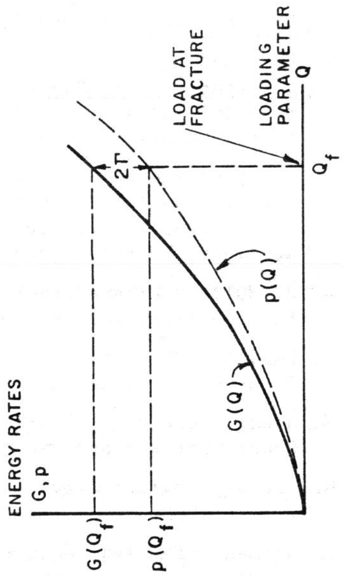



FIG. 4 PLOT OF FRACTURE CRITERION FOR A DUCTILE MATERIAL





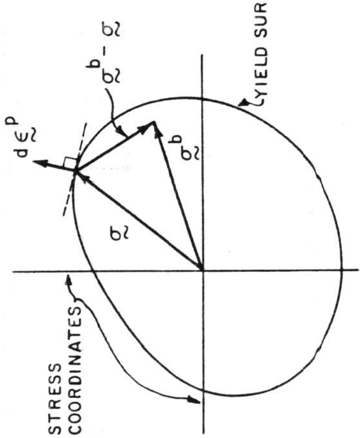



FIG. 5 CONVEX YIELD SURFACE IN STRESS SPACE WITH NORMAL PLASTIC STRAIN INCREMENTS

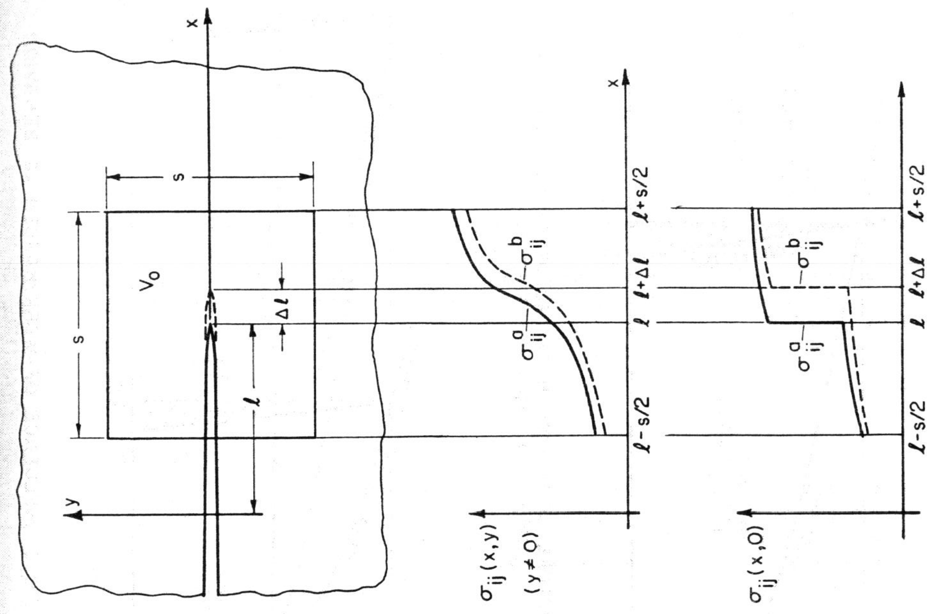



FIG. 6 STRESS CHANGES DUE TO CRACK EXTENSION IN AN ELASTIC PERFECTLY PLASTIC MATERIAL

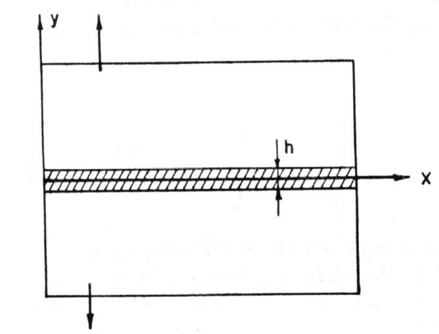



FIG.7 STRIP MODEL

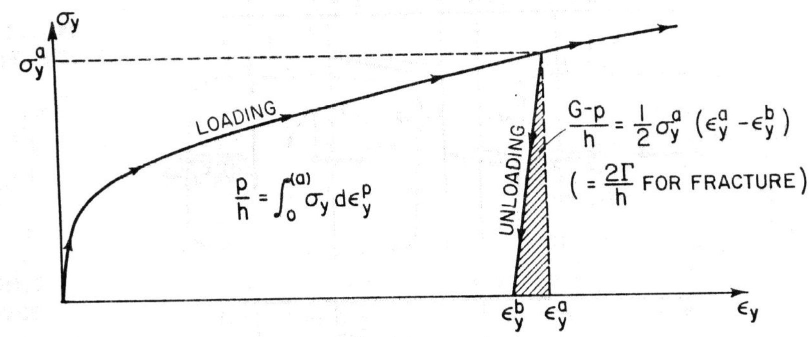
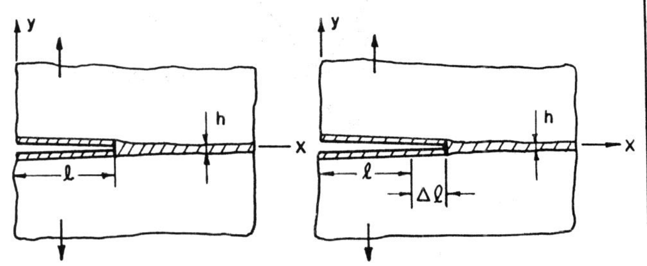



FIG.9 FRACTURE CRITERION IN TERMS OF STRESS - STRAIN CURVE FOR STRIP MATERIAL




(a) INITIAL CRACK

(b) CRACK EXTENDED BY $\Delta \ell$

FIG. 8 CRACK EXTENSION IN STRIP MODEL

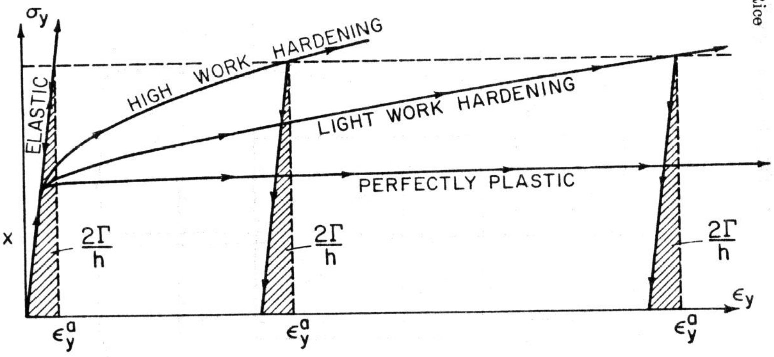



FIG.10 INFLUENCE OF WORK HARDENING BEHAVIOR ON FRACTURE CRITERION