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A Heuristic Approach to the Theory of Fatigue Fracture as Dual to Yielding

Sitiro Minagawa*

Following on from Prof. Kondo's lecture, we intend to give a more detailed explanation of the theory of stress space and its application to the construction of a theory of fatigue fracture. This investigation was started under the guidance of Prof. Kondo and has been carried out as part of his general scheme for a non-Riemannian plasticity theory**.

1. Stress space (7,8,9,10)

The theory of stress space was started by H.Schaefer (11), who introduced a Riemannian space which has the stress functions of three dimensions as the components of the metric tensor, and indicated that the Riemann-Christoffel curvature tensor of the given space is comparable with the stress tensor of the body, although an earlier suggestion of it appeared when E.Beltrami (12) found that the relation between the stress functions and the stress is equivalent in form to the relation between the strains and the incompatibility.

In the geometrical method of approach, we have hitherto used, a Riemannian space having the strains as the components of the metric tensor has been taken into consideration. In the new method of approach, the stress functions of three dimensions are used instead of the strains. Below is a table showing the geometrical terms of the space and the

	strain space	stress space
metric tensor	strain	stress function
R.-C. tensor	incompatibility dislocation pair	stress
torsion tensor	dislocation	moment -stress dual dislocation

*) Associate Professor, Institute for Strength and Fracture of Materials, Tohoku University, Sendai, JAPAN

**)Details of this subject can be found in, e.g., (1,2,3), as well as in the paper by Prof. Kondo immediately preceding the present one (4). Investigations in the elsewhere should be referred to, e.g., (5,6).

physical terms connected with deformation and stress. If we refer the strain tensor to the metric tensor of the space, the Riemann-Christoffel curvature tensor may be compared with the incompatibility. In this case the space is called "the strain space". On the contrary, if we refer the stress functions to the components of the metric tensor, the Riemann-Christoffel curvature tensor of the space reduces to the stress tensor of the body. Such a space we call "the stress-function space", or, simply, "the stress space".

We can, further, obtain a general non-Riemannian stress space by referring Cosserat's moment-stress to the torsion tensor of the space. Such a generalization has been made by R.Stojanovitch (13), S.Amari and K.Kagekawa (14), and by others*.

Bianchi's first and second identities are of such forms

$$\overset{*}{\nabla}_{[m}\Sigma_{lk]j}^{\cdot\cdot\cdot i} = 2\, T_{[ml}^{\cdot\cdot n}\Sigma_{k]nj}^{\cdot\cdot\cdot i} \qquad (1.1)$$

and

$$\Sigma_{[lkj]}^{\cdot\cdot\cdot i} = 2\,\overset{*}{\nabla}_{[l}T_{kj]}^{\cdot\cdot i} - 4\, T_{[lk}^{\cdot\cdot m}\, T_{j]m}^{\cdot\cdot i} , \qquad (1.2)$$

where $\Sigma_{lkj}^{\cdot\cdot\cdot i}$ and $T_{kj}^{\cdot\cdot i}$ are the Riemann-Christoffel curvature tensor and the torsion tensor of the space, and $\overset{*}{\nabla}_i$ means the absolute derivative. Neglecting the terms of higher order, they reduce to

$$\partial_j Z^{ij} = 0 , \qquad (1.3)$$

and

$$Z_{[kj]} = \partial_m \left(T_{kj}^{\cdot\cdot m} + 2\, \delta_{[k}^{m}\, T_{j]} \right) , \qquad (1.4)$$

respectively, where δ_i^j is the Kronnecker delta, and

$$T_i = T_{ji}^{\cdot\cdot j} , \qquad (1.5)$$

$$Z^{ij} = \Sigma^{ij} - \frac{1}{2}\gamma^{ij}\,\Sigma, \qquad (1.6)$$

the Σ^{ij}, Σ and γ^{ij} being the Ricci tensor, the scalar curvature and the contravariant component of the fundamental metric tensor of the space, respectively.

Therefore, if we refer Z^{ij} to the stress tensor of the body, and

*) Dr. Amari will himself give details of Amari-Kagekawa's theory in the lecture following the present one. Now we shall develope the explanation along the line of calculation extracted from (15). In (10) we have also developed the theory along a similar line of thought.

$$M_{kj}^{\cdot\cdot m} = T_{kj}^{\cdot\cdot m} + 2\, \delta_{[k}^{m}\, T_{j]} \qquad (1.7)$$

to Cosserat's moment (or torque-) stress, equations (1.3) and (1.4) are comparable with the fundamental equations of the Cosserat continuum. Therefore, the correspondence such as summalized on the table can be made*.

2. Fatigue fracture as dual yielding (15, 16, 17)

As can be seen from the table, there is a dual correspondence between the stress space and the strain space. That is, the one is derived from the other by the substitution respectively of the stress function and the stress for the strain and the incompatibility, and vice versa. By such a substitution, the formulas which include the terms connected with the stress space are derived from those connected with the strain space, and moreover, from a relation between the stress space variable and that of the strain space, another such relation which is of dual form to the former may be obtained. The problem that confronts us is the determination of physical features of the deformation which are implied by the given dual mathematial formula or relation (see, Fig. 1).

For instance, a stress-strain relation is adopted as a basic mathematical relation to describe the features of the deformation. The dual mathematical relation of it is the incompatibility versus stress function relation. The problem we have to solve is how to determine the deformation formulated by the latter relation.

The concept of the dual yielding is considered initially along the line of thought we have just indicated above. Prof. Kondo's equation of yielding is of such form

$$\partial_l \partial_j \left(B^{ijkl}\, \partial_k \partial_i u \right) - \partial_i \left(\sigma^{ij}\, \partial_j u \right) = 0, \qquad (2.1)$$

and his boundary conditions

$$B^{ijkn}\, \partial_k \partial_i u = 0 \qquad (2.2)$$

and

$$\partial_k \left(B^{injk}\, \partial_j \partial_i u \right) - \sigma^{ni}\, \partial_i u = 0, \qquad (2.3)$$

where B^{ijkl} are the matter constants, σ^{ij} the stresses and n means the direction normal to the surface of the body.

The dual of equation (2.1) must be of the form

*) As regards the dislocation pair, dislocation and dual dislocation, references (1 - 4, 14) should be referred to.

$$\partial_l \partial_j (C^{ijkl} \partial_k \partial_i v) - \partial_i (J^{ij} \partial_j v) = 0, \qquad (2.4)$$

and those of (2.2) and (2.3)

$$C^{ijkn} \partial_k \partial_i v = 0 \qquad (2.5)$$

and

$$\partial_k (C^{injk} \partial_j \partial_i v) - J^{ni} \partial_i v = 0, \qquad (2.6)$$

where C^{ijkl} are the matter constants, J^{ij} the incompatibility and v is the multi-dimensional deflection of the material point which is similar to u in Prof. Kondo's equation.

The appearance of v means that the material point deviates from physical three-dimensional space into the enveloping multi-dimensional space as shown in Fig. 2. Equations (2.4), (2.5) and (2.6) indicate that such a multi-dimensional deflection of the material point appears when the incompatibility reaches a certain critical value.

In the previous lecture (4), Prof. Kondo indicated that those two groups of equations are derived from the principle of minimum energy, one with respect to the variation of the metric of the strain space and another with respect to that of the stress space. He also named the physical counterpart of the latter group of equations as "the dual yielding".

According to Prof. Kondo, the multi-dimensional deflection v is referable to the genesis of the fracture, and the incompatibility J^{ij} to an atmosphere which appears in the interior of the material body when the fatigue is advanced. Through such an interpretation equation (2.4) becomes one which describes a critical phenomenon that The genesis of fracture starts to grow into visible size then the atmosphere reaches a certain critical value. Therefore, the equations of dual yielding are those of fatigue fracture for, at least, a certain idealized type.

That the equations are those of fatigue fracture must be confirmed by the fact that all physical properties of the fatigue fracture of the materials are derived from those equations by means of analytical procedures. One such confirmation procedure is the derivation of the S - N curve, such as will be concerned with in the next section.

3. Fatigue fracture of an isotropic body

If the materials are isotropic, then the material constants C^{ijkl} vanish except for the two of the forms C^{iiii} and C^{iijj} and further those components become respectively independent of the suffixes. Therefore, they may be put into

$$C^{iiii} = C \quad \text{and} \quad C^{iijj} = qC,$$

where C and q are the scalars. In this special case, equations (2.4), (2.5) and (2.6) take such forms as

$$C \, \Delta \, \Delta \, v - J^{ij} \partial_j \partial_i v = 0, \qquad (3.1)$$

$$(1+q) \, C \, \partial_n^2 v - q \, C \, \Delta \, v = 0 \qquad (3.2)$$

and

$$(2+q) \, C \, \partial_n v - (1+q) \, C \, \partial_n^3 v - J^{i}_{\cdot n} \partial_i v = 0 \qquad (3.3)$$

respectively, where Δ means the Laplacean operator. If we assume that the atmosphere which grows in the interior of the material body is isotropic, then the tensor J^{ij} must be

$$J^{ij} = \delta^{ij} J, \qquad (3.4)$$

J being a scalar. Under such an assumption, equation (3.1) can further be reduced to

$$C \, \Delta \, \Delta \, v - J \, \Delta \, v = 0. \qquad (3.5)$$

Next we shall consider the fundamental metric form of the stress space. We assume that, in the microscopic region of the material body, the disturbance of the materials by fatigue is homogeneous and spherically symmetric with respect to an arbitrary point of the body, and is independent of the large scale state of the materials. Under such an assumption, the metric differential form is reduced to

$$d\theta^2 = \frac{1}{1 - sr^2} dr^2 + r^2 (d\alpha^2 + \sin^2\alpha \, d\beta^2), \qquad (3.6)$$

where s is a value connected with the stress amplitude, and r, α and β the cubic coordinates.

From such a metric form and from equation (3.5) we have either

i) $p = 0$ or ii) $s = 0$.

In Fig. 3 we draw a graph which has the values of s and p as the ordinate and abscissa respectively. Since the value of p increases with the time, it may be used instead of the time, or of the number of cycles of stress, whereas the value of s is connected with the amplitude of stress acting on the test piece from the out side. Solution i) means that the test piece is damaged as soon as the incompatibility becomes finite. Solution ii) means that the test piece is damaged as soon as it is loaded. Therefore, in such a case, the S-N curve takes a form as shown by the thick line in the figure.

Of course, the S-N curve is not necessarily of such a simple form, but may be of the form such as shown by the dotted line. Prof. Kondo has shown that such an S-N curve can be derived from the above mentioned equations through a more advanced analysis. He demonstrated this in the previous lecture (4).

4. Concluding summary

Inspired by the very implicative investigation of Prof. Kondo into the multi-dimensional geometrical formulation of the critical condition for yielding, we hit upon the possibility of assuming the same type of multi-dimensional formalism of the critical condition for fatigue fracture. This is anticipated by considering the formal duality between two geometrical formulations of the state of the body; one describing the deformation and the other the stress.

The analytical justifications of the assumption, as well as most of the interpretation of its physics, has been provided by Prof. Kondo who recognized it as a physical possibility.

The equations we assumed (2.4), (2.5) and (2.6) indicate the possibility of existence of a critical phenomenon such that

> when the growing incompatibility reaches a value which is obtained as an eigen value of the field equation, the multi-dimensional deflection perpendicular to the physical 3-space needs to appear.

This being a dual of yielding, is comparable, as Prof. Kondo pointed out, with the process of the fracture of materials by fatigue:

> when a growing atmosphere reaches a certain amount, genesis of fracture begins to develope into visible size.

The comparison is completed by referring the incompatibility to an atmosphere which is created through the process of fatigue, and the multi-dimensional deflection to an unobserved object to be set in correspondence with the genesis of fracture.

What we have undertaken next is a heuristic derivation of a prototype of the S-N curve for the fatigue experiment. For simplicity's sake, a certain isotropy was assumed to arrive at a simplest form of the S-N curve which agrees with the half abscissa and the half ordinate.

The investigation discussed here has just been started and there are many problems to be attacked. With the development of the analysis the theory may have to be in part refined. However, we believe that the ideas we have just considered may indeed offer a new key to the solution of the problem of fatigue fracture.

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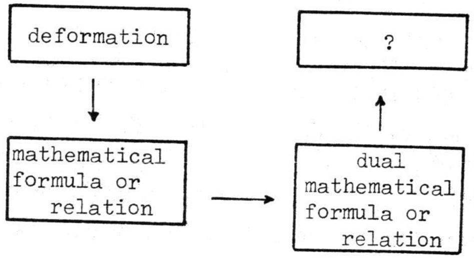

Fig. 1 Deformation and dual deformation

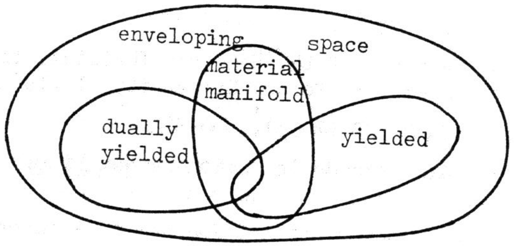

Fig. 2 Liberation and dual liberation

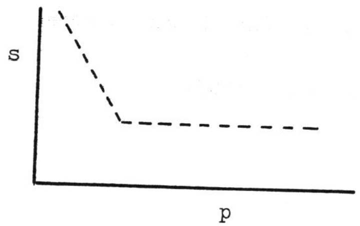

Fig. 3 Typical S - N curve