15. Fractures Produced by Stress Waves

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ABSTRACT

The application of stress wave techniques to the investigation of the mechanism of brittle fracture is described. The "internal strengths" of specimens of polystyrene and polymethyl methacrylate were found to be not appreciably greater than their "surface strengths," whereas in soda glass, internal fractures could not be produced even by large internal tensions. The tensile strength of soda glass for loading times of a few microseconds has been shown to be about twice that for loading times of a few seconds. When ordinary viscous liquids are subjected to large impulsive tensions, the value of the cavitation threshold becomes comparable to the tensile strength of glasslike solids, and for any viscous liquid there is a rate of loading at which brittle fracture takes over from bubble formation.

Introduction

The use of stress waves in the study of fracture goes back nearly a century to the work of John Hopkinson,[1] who measured the strength of metal wires when subjected to tensile impacts. He was able to show that the values of the yield point and of the ultimate strength of iron wires were nearly twice as great, when measured under these conditions, as they were in static tests. His son Bertram Hopkinson [2] continued and extended this work; it was he who first investigated the phenomenon of "scabbing" or "spalling," which occurs when compressive stress pulses are reflected at the free boundaries of a specimen. The period between the two world wars produced few further advances in the subject, but during the last decade the results of a large number of experimental and theoretical investigations in this field have been published both in Europe

and in America.[3–9] This revival of interest in the subject was primarily due to military problems concerned with the resistance of armor to attack, but the work has received continued impetus by the increasing availability of electronic methods of measuring rapidly changing displacements and stresses and by recent advances in the techniques of high-speed photography.

There are three aspects in the study of fracture where stress wave techniques can be used to advantage:

(*a*) The tensile strength of most brittle solids is found to be sensitive to time of loading, and most published data cover loading times with durations between one second and several days or weeks. By studying the strength of materials when subjected to stress pulses of microseconds' duration, the time scale can be extended by four to five decades. Measurements by this means should help to elucidate the reasons for the dependence of the strength on loading time in brittle solids.

(*b*) In the normal tensile testing of brittle solids, a crack develops from a single flaw, and if, for example, a specimen is in a state of simple tension, the value of the tensile strength that is measured corresponds to the "worst" flaw in the specimen and gives no information at all about other flaws that may be present. Under stress wave loading, it is possible for fractures to start simultaneously at a vast number of fracture nuclei, and, since the stress is removed before the fractures can run more than a few millimeters in the specimen, stress wave techniques afford a method of studying flaw distribution in brittle materials.

(*c*) The tensile strength of many brittle materials, such as glass, is found to depend markedly on the condition of the surface, since it is from surface imperfections that running cracks normally start. With some materials, the maximum tensile stress that a specimen can withstand can be greatly increased if the surface is specially treated; for example, freshly drawn glass rods are found to be very much stronger than rods whose surfaces have been exposed to the atmosphere for some time. Now it is clearly desirable to measure the maximum tensile stress that the interior of a specimen can withstand, since this gives an upper limit to the strength that can be achieved by suitable surface treatment. The setting up of internal tensions in a specimen while the surfaces are free from stress is difficult to achieve with static loading. Such stress distributions can, however, be produced by suitable reflection and focusing of stress waves, and, with such techniques, measurements of the *internal strength* of brittle solids can be made.

In the work described in the present chapter, which has been carried out by the author and his colleagues, small explosive charges have been used to produce sharp stress pulses of a few microseconds' duration. The materials investigated have included some glasslike plastics, soda glass, and various liquids covering a wide range of viscosities. The purpose has been to help to elucidate the mechanism of brittle fracture in these materials, as outlined under sections (*a*), (*b*), and (*c*) above, and especially, to study the relation between cavitation in viscous liquids and the incipience of fracture in glasslike solids. These two phenomena have much in common. As first pointed out by Griffith,[10] the measured strength of brittle solids does not depend primarily on molecular cohesive forces but on the existence of tiny flaws or inhomogeneities that grow into cracks under the influence of an applied tensile stress. Because this mechanism, rather than the simultaneous rupture of all the molecular bonds in the fracture plane, is the cause of failure, the observed strength of amorphous solids is a small fraction of the values calculated theoretically from the magnitudes of molecular attractions. On theoretical grounds, liquids too would be expected to withstand considerable tensile stresses.[11] The inability of ordinary liquids to maintain finite tensions is due to the presence of nuclei from which bubbles grow as soon as a negative pressure is applied. With liquids, this can be illustrated by carefully purifying and degassing a sample when, as first shown by Berthelot,[12] it will withstand tension, although, as Temperley [11] has pointed out, the observed values for water still fall far short of those expected theoretically.

Experimental

In general, it is not feasible to produce tensile stress pulses by direct means, except where the specimen is in the form of a wire, and a falling weight or similar device is used to give the end of the wire an impulsive pull. Compressive pulses are much easier to generate and can be produced either by the impact of a high-speed projectile or by the detonation of a small quantity of explosive. The reflection of a compression pulse at a free boundary of a specimen results in a tension pulse being propagated back into the specimen, and this phenomenon has been used in the work described here for studying fracture and cavitation.

The reflection of a compression pulse that is approaching a free surface at normal incidence is shown diagrammatically in Fig. 1. The reflected pulse is one of tension, and the resultant stress in the neighborhood of the surface is obtained by adding together the stresses resulting from the incident and reflected pulses. These are denoted by the thinnest lines in the figure, and the resultant stress is denoted by the thickest line, while the broken line corresponds to that portion of the pulse that has already been reflected. Six stages in the reflection are shown, and it may be seen that, as reflection proceeds, the resultant stress changes from com-

pression to tension, the value of the tensile stress increasing away from the free surface. The maximum tensile stress, which is equal to the maximum compressive stress in the incident pulse, first occurs at a distance less than, or equal to, half the pulse length from the surface. For a steep-

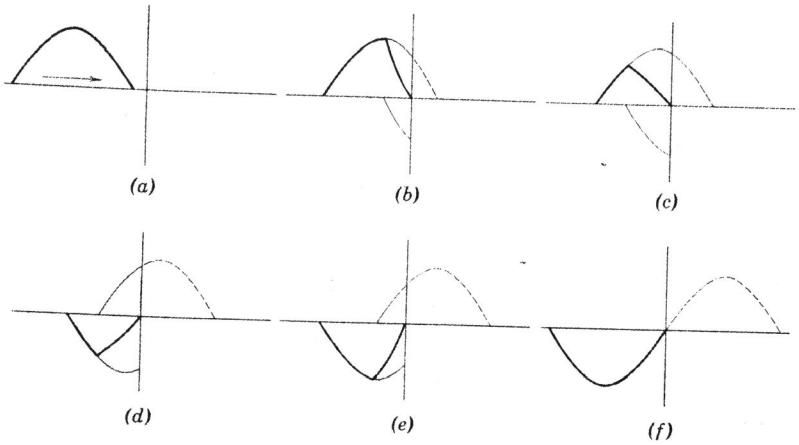

Fig. 1. Reflection of a pulse at a free boundary.

fronted pulse, it occurs at exactly half the pulse length, while for a symmetrical pulse, such as that shown in the figure, it occurs at a quarter of the pulse length (Fig. 1*e*).

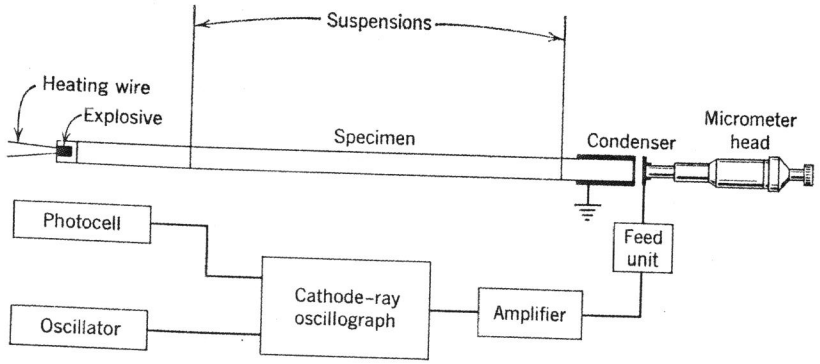

Fig. 2. Experimental arrangement.

The simplest geometrical arrangement in which such a tension pulse can be produced experimentally is that of a blow applied to one end of a long rod; Fig. 2 shows an experimental arrangement with which an investigation has been made of this effect on rods of glass and plastics. With this arrangement, the specimens, which were between $\frac{1}{4}$ in. and

1 in. in diameter, were freely suspended by long thin threads, and the "blow" was applied by the detonation of a small charge of lead azide contained in a recess in a short, cylindrical, plastic anvil of the same diameter as the rod. The anvil was wrung onto one end of the rod with a thin layer of grease; the explosive charge could be detonated by means of an electrically heated wire. The opposite end of the rod, where the compression pulse is reflected, was coated with a graphite layer grounded by means of a thin wire. The graphited end of the rod formed the grounded plate of a condenser unit, the other plate being an insulated metal disk that was attached to a micrometer head. The insulated plate was charged to 540 volts through a 50-megohm resistance, and the change in potential of the plate was proportional to the displacement of the end of the rod, which was produced by the reflection of the pulse. These changes of potential were recorded with a high-speed cathode-ray oscillograph. The electrical circuits were similar to those previously used by Davies [13] and by the author.[14,15]

If the displacement of the end of the bar is u and the stress in the incident compression pulse is σ, then, for a nondispersive medium,[8]

$$\sigma = \frac{1}{2}\rho c \frac{du}{dt} \tag{1}$$

where ρ is the density of the rod and c is the velocity of extensional waves in it ($c = (E/\rho)^{1/2}$ where E is Young's modulus). Consequently, from the displacement-time curve for the end of the rod, the shape of the compression pulse arriving at the end could be calculated. In practice, this involves a numerical differentiation of the displacement-time curve obtained from the oscillograph record.

When the incident compression pulse was of sufficiently large amplitude, it produced a tensile fracture in the rod after reflection, and by variation in the size of the explosive charge, the critical stress amplitude at which the rod broke could be determined. Figure 3 shows a photograph of two $\frac{1}{2}$-in. diameter polystyrene rods. The upper specimen was produced with an explosive charge of 0.16 g of lead azide, and the rod was broken completely. In the lower specimen, a slightly smaller charge was used, and an *internal fracture*, which does not extend to the free surface, was formed. The significance of this internal fracture will be discussed later.

Experiments were carried out [16] with rod specimens of soda glass, polystyrene, and Plexiglas (polymethyl methacrylate). Similar specimens were tested statically. A tensile testing machine was used on the plastic specimens, while the glass rods were placed horizontally on two cylindrical supports and loaded at the middle. Table 1 compares typical results of the values of tensile strength obtained from stress wave measurements

with those values found statically. For all three materials, the stress pulses were found to be nearly symmetrical in shape and approximately of the form shown in Fig. 1. When the charge was just sufficient to produce a fracture, this occurred at about one quarter of the pulse length

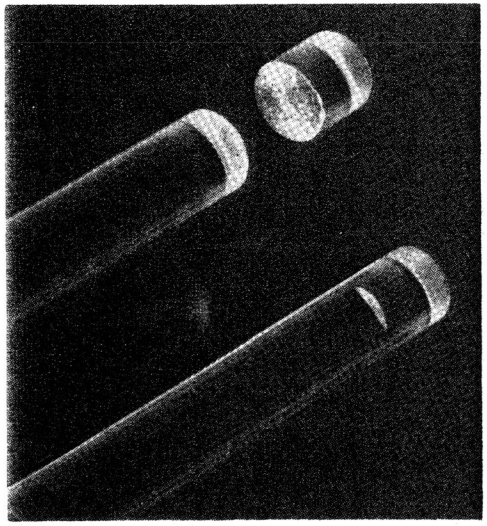

Fig. 3. Fractures in $\frac{1}{2}$-in. diam polystyrene rods.

away from the free end, as predicted by theory. (For example, with the $\frac{1}{4}$-in. glass rods, the fracture was 0.25 in. from the end. The velocity c in the rod was found to be 16,500 fps, and the effective pulse duration was about 6 μsec.)

TABLE 1. Static and Dynamic Tensile Strengths

	Glass	Polystyrene	Plexiglas
Rod Diameter (in.)	0.25	0.5	0.5
Static Strength (psi)	11,500	5,500	6,200
Dynamic Strength (psi)	25,000	10,500	16,000
Lead Azide Charge (mg)	20	160	150
Pulse Duration (μsec)	6	25	20
Internal Fracture	No	Yes	Yes

For slightly bigger charges, the fracture occurred closer to the end, and with large enough charges, multiple fractures were produced. Thus, with a polystyrene rod 6 in. in length, a 200-mg charge produced three parallel fracture surfaces. These secondary fractures result from the reflection of the tail of the pulse at the new surface produced by the first fracture. This phenomenon of "multiple scabbing" has been discussed by Rinehart and Pearson.[9]

The results obtained from experiments of this type are of value for the reasons given in the Introduction: Namely, they give values of the tensile strength for loading times that are shorter by several orders of magnitude than those obtained hitherto. As can be seen from Table 1, the strength of all three materials investigated was about twice as great when specimens were loaded for a few microseconds as when loaded for a few seconds.

Measurements of the dependence of the tensile strength of soda glass on the time for which a load is applied were first carried out by Grenet [17] at the end of the last century. Grenet found that the strength of glass rods for a loading time of 40 hr was about one-half that for a loading time of 1 sec. It is perhaps significant that the ratio between these loading times is of the same order as that given in Table 1 and that the ratio of the strengths is also about the same. It has been suggested that the lower strength at the longer times of loading occurs because the Griffith flaws on the surface of the glass are opened by the applied forces. This allows the humidity in the atmosphere to reach the end of the flaw and so reduces the value of the specific surface energy on which the growth of the crack depends. If the surface is carefully dried and the tensile tests carried out in a vacuum, it is possible to decrease the time dependence, and the strength for all loading times can be made higher than that for loading times of the order of a second.

If the above explanation is correct, the values obtained with the microsecond stress-pulses might be expected to correspond to the upper limit of strength that can be achieved by reducing loading times, in that no appreciable increase in surface contamination is likely to occur in these short-term experiments.

With the plastic specimens, the situation is more complex. These materials are viscoelastic, and their mechanical properties are consequently highly dependent on the rate of loading. Measurements of the dependence of the elastic modulus on the frequency of an applied sinusoidal force for such plastics have been carried out by Lethersich,[18] and the author [14,19] has investigated their dynamic behavior by wave-propagation methods. These results show that for polymethyl methacrylate (Plexiglas) the value of Young's modulus is more than doubled by a change in the time scale that corresponds to the change in value noted in Table 1 between static and dynamic tests. Thus, the *strain* at fracture is not very different in the static and the dynamic measurements; indeed, this difference between the value of the static and dynamic strength of this material may possibly be accounted for entirely by the change in

the effective value of the modulus. The change of modulus with rate of loading is considerably smaller for polystyrene than for Plexiglas and is of the order of a 40% increase in going from the static to the dynamic tests. It should perhaps be remarked that the fractional increase in strength of this material as shown in Table 1 is also smaller than it is for Plexiglas.

As mentioned earlier, it was found that internal fractures could be produced in polystyrene and Plexiglas specimens for a narrow range of stress amplitudes. These fractures were in the form of circular disks whose centers were on the axis of the bar. Such a fracture can be seen in the lower rod shown in Fig. 3. The reason for such internal fractures is almost certainly the slightly higher tensile stress in the region around the axis of the rod. This higher stress is due to the effect of the lateral inertia of the rod at these very high rates of loading. It may be shown from the exact theory of wave propagation of elastic waves in cylindrical bars [20] that the stress distribution produced by an extensional wave traveling in a rod of finite diameter is not uniform. Davies [13] has carried out the mathematical analysis for sinusoidal waves in a bar of material having a Poisson ratio of 0.29. In this case, when the wavelength is five times the radius of the bar, the stress component in the direction of the axis of the bar is about 30% greater at the axis than it is at the surface. In order to determine the corresponding difference for a stress pulse, it would be necessary to consider the stress distribution of the individual Fourier components of the pulse, and then to integrate. Qualitatively, however, it may be estimated that for a smooth pulse of 25-μsec duration that is traveling in a $\frac{1}{2}$-in. diameter bar of polystyrene (as was used for the experiments listed in Table 1) the difference between the values of the stress at the axis and at the surface of the bar will be of the same order, although somewhat less than that calculated by Davies. Thus the tensile stress necessary to produce an internal fracture in this material is not more than 30% greater than that required to start a fracture at the surface. Similar considerations are found to apply to Plexiglas, although the higher attenuation of this material to high-frequency waves somewhat complicates the problem.

With rod specimens of glass, no such internal fractures could be produced. This is to be expected, since in static tensile tests on glass rods the strength is found to be highly sensitive to the condition of the surface, which shows that the internal strength must be much higher than the surface strength.

For plane stress waves traveling down long rods, the tensile stress produced at the axis is only of the order of 30% in excess of that at the surface. Very much greater differences are developed, however, when

a small explosive charge is detonated at the center of one of the end faces of a cylinder of larger diameter. Under these conditions, a spherical compression pulse diverges from the charge and is reflected back by the cylindrical surface as a wave of tension. This reflected wave converges onto the axis producing tensile stresses in the region that are many times greater than any that are built up on the free surface of the cylinder. Figure 4 shows a series of six cylindrical specimens of polymethyl methacrylate that have had 120-mg charges of lead azide detonated on the center of each of their top faces. The specimens were 1 in. in diameter, and their lengths increased by half-inch steps from $\frac{1}{2}$ in. to 3 in. It may

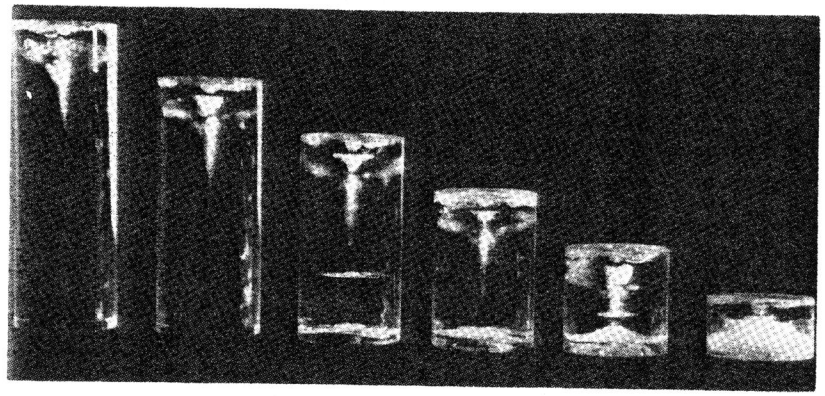

Fig. 4. Fractures in 1-in. diam polymethyl methacrylate cylinders.

be seen from the photographs how the focusing of the pulse reflected from the cylindrical surface has produced intense fracturing along the axis of the cylinder, and it should be noted that the region contains a vast number of tiny fractures which have started independently as a result of the large tensile-stress build-up. It may be seen that, in the cylinder 2 in. in length, an internal, flat, disk-shaped fracture has also been produced; this is of similar origin to the fracture shown in Fig. 3 of a polystyrene specimen. In a similar set of experiments with glass specimens, axial fracturing, such as that shown in Fig. 4, could not be produced. This indicates that the internal strength was much greater than the surface strength. It would appear, however, that only by such focusing techniques will it be possible to measure the internal strength of glass, as this is likely to be of the order of millions of pounds per square inch.

Cavitation Experiments

As mentioned earlier, ordinary liquids will not withstand appreciable tensile stresses if these are applied statically. Transient tensions having

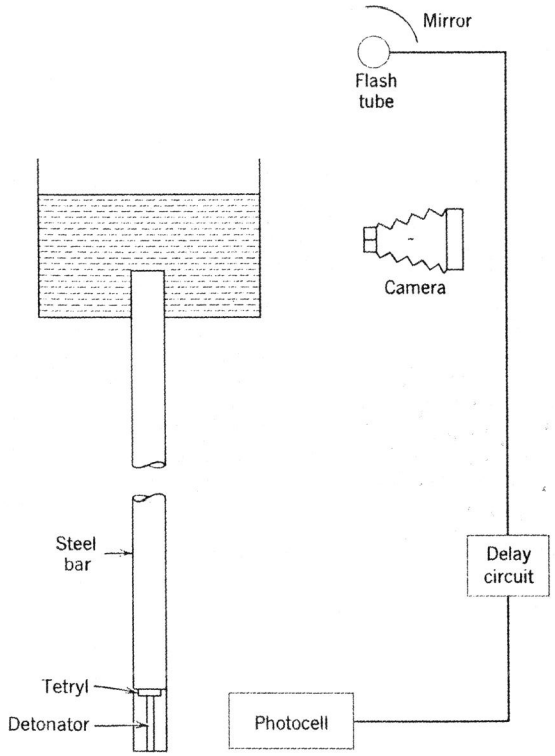

Fig. 5. Apparatus for observing cavitation.

durations of only a few microseconds can, however, be applied to ordinary liquids without cavitation occurring. This was shown in some earlier work [21] on underwater explosions, where it was found that when an explosive charge is detonated at a depth below the surface greater than a certain value called the "critical depth," the surface remains unbroken even though the top layers of the water have had to sustain quite large momentary tensions. (The critical depth for a 1-lb charge of P.E.T.N. is found to be about 30 ft.) From high-speed photographs of the splashes produced at lesser depths, it was shown that untreated water could, under these conditions, sustain a tension of several hundred pounds per square inch without breaking up.

In order to investigate similar phenomena on a laboratory scale, the apparatus that is shown diagrammatically in Fig. 5 was employed. The stress pulse is here communicated to the liquid by means of a 1-in. diameter steel bar, at the lower end of which an explosive charge is detonated. The liquid is contained in a rectangular metal tank with two parallel glass sides, and the steel bar enters the bottom of the tank through a soft gland and protrudes about 2 in. into the tank. The liquid surface was arranged to be 1 to 2 in. above the top end of the bar. Photographs were taken, using a flash tube, which gave an effective exposure time of about 1 μsec. The flash was arranged to occur a few microseconds after the stress pulse had been reflected at the free surface of the liquid. Figure 6 shows a photograph taken for distilled water. It may

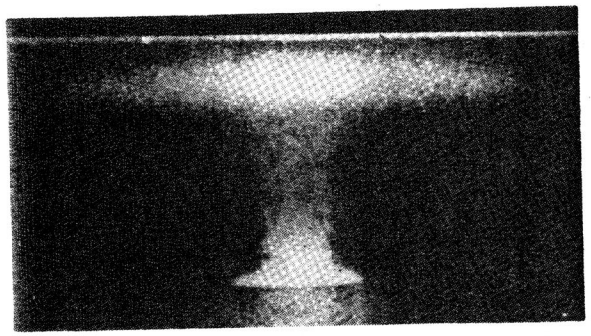

Fig. 6. Cavitation produced by reflected stress pulse in water.

be seen that the cavitation region is in the form of a horizontal disk, the center of which is about 1 in. below the water surface. (The narrow vertical column of cavitation rising from the tank is due to the lack of adhesion of the water to the top surface of steel. This could be seen from photographs taken before the main cavitation region had sufficient time to develop.) The fact that there is a region of unbroken water between the top of the disk of cavitation and the water surface shows that the liquid has here been able to withstand the quite considerable tensile stresses to which it has been submitted. This region of unbroken water increases in depth as the distance from the axis of the bar increases. This is because the incident pulse is weaker here, and the reflected tension pulse has to travel farther back through the tail of the incident compression pulse before the critical value of tension required to produce bubble growth is reached (Fig. 1).

These experiments show the close similarity between cavitation and fracture produced by stress pulses. The work has been continued [22,23]

by the author's colleague, Dr. T. H. Bull, who studied the behavior of more viscous liquids and made measurements of the maximum tensile stress that liquids could withstand under these conditions. Bull used a cylindrical water column of the same diameter as the bar, so that the compression pulse that was propagated along the column was plane rather than divergent; he was thus able to measure the maximum amplitude of the reflected tension pulse that could travel back through various liquids. The results of his measurements are shown in Fig. 7, which gives a logarithmic plot of maximum tensile stress (in dynes/cm²) P_c against the logarithm of the viscosity η. The liquid of lowest viscosity was tap

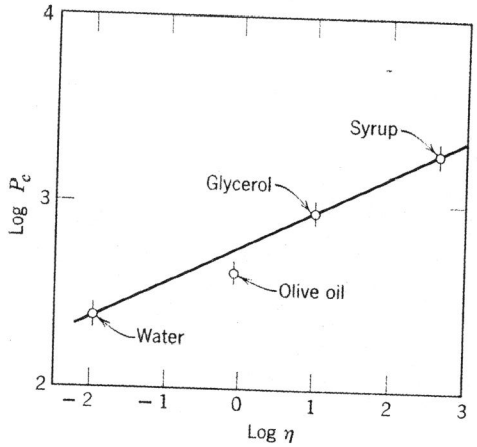

Fig. 7. Log-log plot of cavitation threshold against viscosity. (After Bull.[23])

water, which was able to sustain a tension of 250 psi, and that of highest viscosity a sugar syrup, which could sustain 2000 psi.

These measurements indicate that, under stress-wave loading, viscosity is the governing factor in determining the maximum tension that untreated viscous liquids can withstand. As can be seen from Fig. 7, the log-log plot of tensile strength against viscosity is approximately linear, and Bull has fitted the empirical relation

$$P_c = K\eta^{0.2} \tag{2}$$

to his results. Here K is a constant that has the value of 3×10^7 in cgs units, and its value will clearly depend on the duration of the stress pulse. In Bull's experiments, this duration was about 30 μsec.

A theoretical treatment of the growth of a spherical cavity in a viscous fluid has been given by Poritsky.[24] Unfortunately, the problem is very complex, and a quantitative evaluation of bubble growth under a given

stress pulse would involve considerable computational difficulties. However, it may be shown that, in the initial stages of growth, the viscous forces, rather than those resulting from inertia or surface tension, are likely to govern the expansion of small bubbles.

It should be noted that the value of the dynamic tensile strength of the sugar syrup (which has a viscosity of 400 poise) is comparable with the internal strength of the plastics shown in Table 1, and, if Eq. 2 also applies at much higher viscosities, the cavitation threshold P_c will closely approach and possibly exceed what is normally considered the tensile strength of the material. Thus, the distinction between cavitation and fracture under stress-wave loading would appear to become somewhat ill-defined under such conditions. The essential difference between the two phenomena is associated with the possibility of the formation of a running crack. An intrinsic property of a perfect fluid is that, at any point, the stress has the same value in all directions, and consequently, on symmetry grounds, any bubble must grow spherically when tension is applied to the fluid. For a running crack to develop, the stress around such a flaw must not be purely hydrostatic, and the medium must be able to sustain shear stress. In viscous fluids, such shear stresses can be built up but will relax exponentially at a rate governed by the *relaxation time* τ. The value of τ is given by the ratio η/G, where G is the shear modulus of the material. The shear modulus of various very viscous liquids has been measured by Benbow.[25] The values he has obtained are of the order 3×10^9 dynes/cm². Thus, for example, if the viscosity is 400 poise (as in the case of the sugar syrup), the value of τ will be about $\frac{1}{10}$ μsec, and, even under ordinary stress-wave loading, running cracks cannot be produced.

Certain organic compounds of definite chemical structure have very much higher viscosities, and it was considered of interest to see how these materials respond to intense stress pulses.[26] One such material is 2′-hydroxy–2:4:4:6:5′-pentamethylflavan, which has a viscosity of 10^{10} poise at 10°C. Its viscosity decreases rapidly with increasing temperature, falling to 500 poise at 50°C. When small lead azide charges were detonated in contact with specimens of this material, a large number of running cracks were produced, although these slowly healed as a result of viscous flow. Figure 8 shows the appearance of a cylindrical disk of this material at different times after a charge had been detonated at the center of the top surface. The temperature was 20°C, when η is about 3×10^7 poise for this material. The disk was 4 cm in diameter and 1 cm thick, and the explosive charge used was 10 mg of lead azide. It may be seen that, immediately after the explosion, the whole volume of the specimen is filled with fractures, a few of which are axial cracks

produced by hoop tensions. The later photographs in Fig. 8 show how the specimen heals, and how, after 5 hr, it has returned almost to its unfractured condition.

Running cracks could not be induced in pentamethylflavan at 20°C by static loading, and this is to be expected, since its relaxation time τ at this temperature is about 10 millisec. This means that any shear stresses induced will relax to $1/e$ of their initial value in this period.

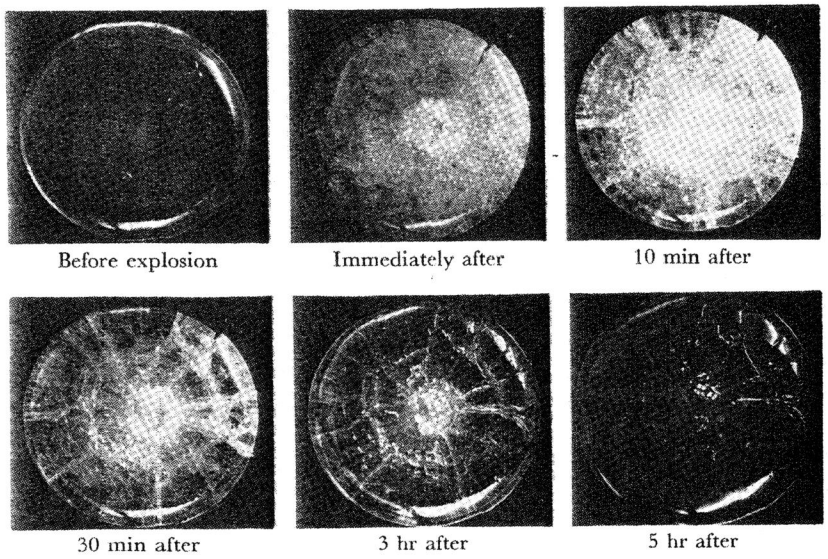

Fig. 8. Healing of fractures produced by 0.01-g lead azide charge in organic glass specimen.

In conclusion, it is perhaps of interest to consider how rapidly a stress would have to be applied to water in order to produce a running crack. If we assume that its shear modulus G has the same order of magnitude as that of the more viscous liquids, the value of τ will be about 10^{-11} sec, and only by the use of stresses that can build up in times of this order could one hope to produce brittle fractures in the liquid.

Conclusion

The use of stress pulses for the investigation of the mechanism of brittle fracture has been discussed, and it has been shown how experiments carried out by this technique can give information about a number of aspects of the subject.

In particular, it has been shown that the internal strengths of the two

plastics studied are not appreciably higher than their strengths at the free surfaces. For soda glass, the strength at microsecond loading has been found to be about twice that for loading times of a few seconds; thus the tendency of the strength to increase with a decrease in the time of application of the stress has been shown to continue over several more decades in the time scale. The fact that internal fractures could not be produced in glass, even when very large internal tensions had been built up, confirms the fact that the internal strength of this material is very large. Further work is required, however, to obtain quantitative measurements of the maximum internal tension that glass will withstand.

The phenomenon of cavitation in viscous liquids has also been discussed, and it has been shown that, for microsecond loading, such liquids can withstand considerable tensile stresses. It would appear that, at viscosities of the order of 10,000 poise, cavitation changes to fracture, in that the cavitation threshold becomes comparable with the stress required to produce running cracks, the governing factor being the relaxation time of the liquid.

The work described has been, to a large extent, exploratory, and it is clear that more experiments are required if a better understanding of the various mechanisms is to be obtained.

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