



Theoretical model of homogenized piezoelectric materials with small non-collinear periodic cracks

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ABSTRACT. An analytical model for the homogenization of a piezoelectric material with small periodic fissures is proposed on the basis of the method of asymptotic expansions for the elastic displacement, the electric scalar potential and the test functions. Starting from the variational formulation of the three-dimensional problem of linear piezoelectricity, we have at first obtained that concerning a cracked piezoelectric structure, before the implementation of homogenized equations for a piezoelectric structure with a periodic distribution of cracks. It then follows, the characterization of the homogenized law between the mechanical strain and the electric potential, on one hand, and the mechanical stress and the electric displacement, on the other hand. Contrary to the previous investigations, the focus of this paper is the development of a mathematical model taking the non-parallelism of cracks into account.

KEYWORDS. Piezoelectric material; Asymptotic expansions; Homogenization; Variational formulation; Periodic cracks.



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INTRODUCTION

The piezoelectric materials are used in an increasing way in technological applications [1-3]. Among the numerous problems which can arise, there is that concerning the global estimation of the homogenized characteristics of non-homogeneous materials, in particular those presenting a periodic distribution of singularities. Significant efforts had been made to the study of periodical cracks in linear piezoelectricity, through an extension to piezoelectric materials of the modeling of periodic cracks in elastic materials [4, 5]. Gao and al.[6] studied, in terms of the Parton assumption and Stroh formalism, the problem of a half-infinite crack in piezoelectric media with periodic crack; reducing it to Hilbert one and getting therefore the closed-form solutions in the media and inside the cracks. Wang and al. [7] provided a theoretical treatment of the dynamic interaction between cracks in a piezoelectric medium under anti-plane mechanical and in-plane electrical incident wave. They used Fourier transform to study the dynamic electromechanical behavior of a single crack, and solved the obtained singular integral equations by Chebyshev polynomials. The single crack solution was then implemented into a pseudo-incident wave method to account for the interaction between cracks.



Somme years more-late, Han and al. [8] obtained the development of a mathematical model to predict the length scale for the spacing of transverse cracks forming in a piezoelectric material subjected to a coupled electro-mechanical external loading condition. In particular, they analyzed the interactions of a row of cracks periodically located in a piezoelectric material layer. Although, one of the remaining problems that need to be treated is that of a periodic array of non-collinear cracks. So, the present paper provides a theoretical model of homogenized piezoelectric materials with small non-collinear periodic cracks through an extension of previous works [9] and [10]. It is organized as follows: Section 2 describes the variational formulation for the three-dimensional problem of linear piezoelectricity. Section 3 develops a variational formulation for the problem of a fissured piezoelectric structure. In Section 4, are presented the homogenized problem of a piezoelectric material with small periodic cracks. Section 5 is then devoted to the formulation of the homogenized local problem in the homogenization period. The analysis of the relationship between the strain and the electric potential on one hand, and the stress and the electric field secondly, is presented in Section 6, just above the conclusion.

VARIATIONAL FORMULATION FOR THE THREE-DIMENSIONAL PROBLEM OF LINEAR PIEZOELECTRICITY

Let Ω be an open connected domain of \mathbb{R}^3 with smooth boundary $\partial\Omega$ made of two parts $\partial_1\Omega$ and $\partial_2\Omega$ in the mechanical sense, and of $\partial_3\Omega$ and $\partial_4\Omega$ in the electrical one. These parts of $\partial\Omega$ represent portions of regular surfaces with smooth common boundary, respectively. Moreover, Ω may be divided into two parts by a smooth surface Γ_S .

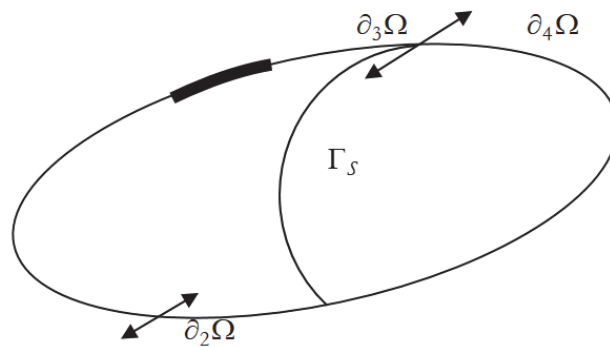


Figure 1: Representation of the open Ω .

In the framework of linear piezoelectricity, the elastic and electric effects are coupled by the constitutive equations:

$$\sigma_{ij} = a_{ijkl} \frac{\partial u_k}{\partial x_j} - e_{kij} E_k \tag{1}$$

$$D_i = \epsilon_{ij} E_j + e_{ikl} \frac{\partial u_k}{\partial x_l} \tag{2}$$

where $u = \{u_i\}$ is the elastic displacement, $\sigma = \{\sigma_{ij}\}$ is the symmetric stress tensor, $E = \{E_i\}$ is the electric field vector, and $D = \{D_i\}$ is the electric displacement vector, with $(i, j, k, l) = (1, 2, 3)$.

We now assume that the elastic coefficients at zero elastic field a_{ijkl} , the piezoelectric coefficients e_{kij} and the dielectric constants ϵ_{ij} at vanishing strain satisfy the following symmetry and positivity properties:

$$a_{ijkl} = a_{jikl} = a_{klij}; e_{kij} = e_{kji}; \epsilon_{ij} = \epsilon_{ji} \tag{3}$$

$$\forall e_{ij}, e_{ij} = e_{ji}, \exists \alpha_0 > 0: a_{ijkl} e_{kl} \bar{e}_{ij} \geq e_{ij} \bar{e}_{ji} \tag{4}$$



$$\forall \xi_i, \exists \alpha_1 > 0 : \varepsilon_{ij} \xi_j \bar{\xi}_i \geq \alpha_1 \xi_i \bar{\xi}_i \quad (5)$$

where α_1 and α_0 are constants; a superimposed bar denoting the complex conjugate. But, under the quasi-electrostatic approximation [9], there exists an electric scalar potential ϕ such that:

$$E_i = -\frac{\partial \phi}{\partial x_i} \quad (6)$$

Moreover, if t_j^0, u_j^0, w^0 and ϕ^0 are prescribed values per unit area, the mechanical boundary conditions can be written as:

$$n_j \sigma_{ij} = t_i^0 \quad \text{on} \quad \partial_1 \Omega \quad (7)$$

$$u_j = u_j^0 \quad \text{on} \quad \partial_2 \Omega \quad (8)$$

and the electric ones are in the following forms:

$$n_i D_i = -w^0 \quad \text{on} \quad \partial_3 \Omega \quad (9)$$

$$\phi = 0 \quad \text{on} \quad \partial_4 \Omega \quad (10)$$

The piezoelectric plate is supposed to be clamped by $\partial_2 \Omega$; and n represents the unit outer normal to $\partial \Omega$. If the body force and extrinsic bulk charge are assumed to be negligible, σ and D are divergence-free, i.e.

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \quad \text{in} \quad \Omega \quad (11)$$

$$\frac{\partial D_i}{\partial x_i} = 0 \quad \text{in} \quad \Omega \quad (12)$$

On the other hand, we have:

$$n_i (\sigma_{ij}^+ - \sigma_{ij}^-) = 0 \quad \text{on} \quad \Gamma_S \quad (13)$$

$$(u_i^+ - u_i^-) = 0 \quad \text{on} \quad \Gamma_S \quad (14)$$

$$(\phi^+ - \phi^-) = 0 \quad \text{on} \quad \Gamma_S \quad (15)$$

$$n_i (D_i^+ - D_i^-) = 0 \quad \text{on} \quad \Gamma_S \quad (16)$$

The variational problem (VP) corresponding to Eqs. (7) to (12) is obtained by introducing the following spaces:

$$V_2 = \left\{ u; u_i \in H^1(\Omega), u_i|_{\partial_2 \Omega} = u_i^0 \right\} \quad (17)$$



$$V_4 = \left\{ \phi; \phi \in H^1(\Omega), \phi|_{\partial_4\Omega} = 0 \right\} \tag{18}$$

Problem (VP): Find (u, ϕ) in $V_2 \times V_4$ such that:

$$\begin{cases} a(u, v) + b(\phi, v) = \int_{\partial_1\Omega} t^0 \cdot v \, ds & \forall v \in V_2 \\ c(\phi, \varphi) + d(u, \varphi) = \int_{\partial_4\Omega} w^0 \varphi \, ds & \forall \varphi \in V_4 \end{cases} \tag{19}$$

where

$$a(u, v) = \int_{\Omega} a_{ijkl} e_{kl}(u) e_{ij}(v) \, dv \tag{20}$$

$$b(\phi, v) = \int_{\Omega} e_{ijk} \frac{\partial \phi}{\partial x_k} \frac{\partial v_i}{\partial x_j} \, dv \tag{21}$$

$$c(\phi, \varphi) = \int_{\Omega} \varepsilon_{ij} \frac{\partial \phi}{\partial x_j} \frac{\partial \varphi}{\partial x_i} \, dv \tag{22}$$

$$d(u, \varphi) = - \int_{\Omega} e_{ikl} \frac{\partial u_k}{\partial x_l} \frac{\partial \varphi}{\partial x_i} \, dv \tag{23}$$

Proposition1. Problem (VP) is equivalent to Eqs. (7) to (12).

Proof. (19)₁ is obtained by multiplying (11) par a test function v_i and by integrating by parts; taking into account the boundary conditions (7) and (8). By analogy, we obtain (19)₂, by multiplying (12) par a test function φ and by integrating by parts; taking into account the boundary conditions (9) and (10). The coefficients a_{ijkl} are assumed to be continuous on Γ_S . For the existence and uniqueness of the solution of problem (VP), see [9].

VARIATIONAL FORMULATION FOR THE PROBLEM OF A FISSURED PIEZOELECTRIC STRUCTURE

We now consider a piezoelectric structure containing a closed crack C , i.e.

$$C = \bar{C} \tag{24}$$

where \bar{C} is the closure of C , and where ∂C is assumed to be smooth. Let us introduce the open subset Ω_C , verifying:

$$\Omega_C = \Omega - C \tag{25}$$

The local equations of linear piezoelectricity for a fissured piezoelectric structure can then be written as follows [10]:

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \quad \text{in } \Omega_C \tag{26}$$

$$\frac{\partial D_i}{\partial x_i} = 0 \quad \text{in } \Omega_C \tag{27}$$



$$u_i = 0 \text{ on } \partial\Omega \tag{28}$$

$$[u_i N_i] \geq 0 \text{ on } C \tag{29}$$

$$\phi = 0 \text{ on } \partial\Omega \tag{30}$$

$$[\phi] \geq 0 \text{ on } C \tag{31}$$

$$(a_{ijkl} u_{k,l} n_j - e_{kij} E_k n_j) \Big|_1 = (a_{ijkl} u_{k,l} n_j - e_{kij} E_k n_j) \Big|_2 = -(a_{NNkl} u_{k,l} - e_{kNN} E_k) \Big|_1 N_i \text{ on } C \tag{32}$$

$$a_{NNkl} u_{k,l} - e_{kNN} E_k \leq 0 \text{ on } C \tag{33}$$

$$D_i n_i \Big|_1 = -D_i n_i \Big|_2 \leq 0 \text{ on } C \tag{34}$$

where N and n represent the unit normal to C , outer to its side 1, see (Fig. 2), and the unit outer normal to the open domain $\Omega_C = \Omega - C$, respectively.

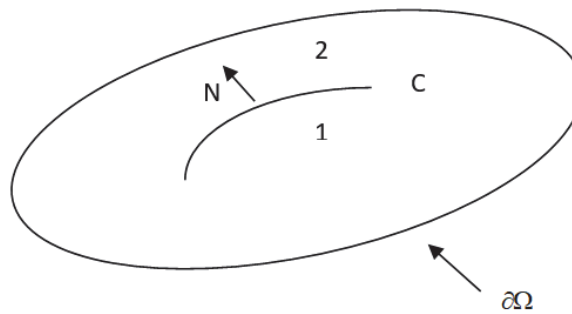


Figure 2: Representation of the fissured piezoelectric structure Ω_C .

These relations express a compression on C according to (33), as well as the normality of the force which acts; with an opposition between the action and the reaction, according to (32). Consequently, the variational formulation (FVP) for the problem of such a fissured piezoelectric structure can be stated as follows:

Problem (FVP): Find (u, ϕ) in $V^{*u} \times V^{*\phi}$ such that:

$$\begin{cases} a(u, v - u) + b(\phi, v - u) \geq 0 & \forall v \in V^{*u} \\ c(\phi, \varphi - \phi) + d(u, \varphi - \phi) \geq 0 & \forall \varphi \in V^{*\phi} \end{cases} \tag{35}$$

where

$$V^u = \{u = (u_i); u_i \in H^1(\Omega_C); u_i|_{\partial\Omega} = 0\} \tag{36}$$

$$V^{*u} = \{u = (u_i); u_i \in V^u; [u_i N_i] \geq 0\} \tag{37}$$

$$V^\phi = \{\phi; \phi \in H^1(\Omega_C); \phi|_{\partial\Omega} = 0\} \tag{38}$$



$$V^{*\phi} = \{ \phi; \phi \in V^\phi; [\phi] \geq 0 \} \tag{39}$$

And where a, b, c, and d are bilinear forms on V^u , $V^{*\phi} \times V^{*u}$, $V^{*\phi}$, and $V^{*u} \times V^{*\phi}$ respectively.

Proposition2. Problem (FVP) is equivalent to Eqs. (26) to (34).

The proof is analogous to that of proposition 1, by taking into account Eqs. (29) and (31).

HOMOGENIZED EQUATIONS-FORMAL EXPANSION

We now consider a linear piezoelectric plate with a ε -periodic distribution of fissures, so that, the period Y of \mathbf{R}^3 , admits a smooth fissure C verifying:

$$C \cap \partial Y = \emptyset \tag{40}$$

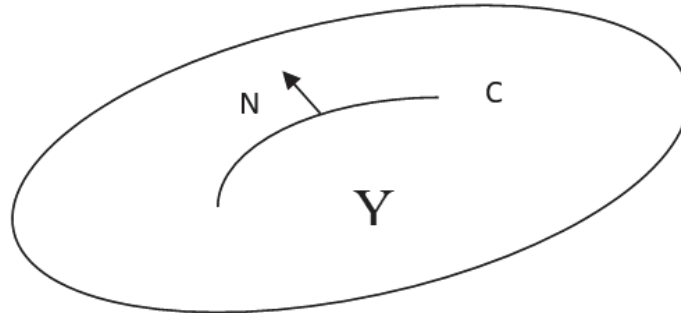


Figure 3: Representation of the period Y with a smooth fissure C .

The fissured material denoted by $\Omega_{C\varepsilon}$ is then defined as follows:

$$\Omega_{C\varepsilon} = \Omega \cap \left\{ x = (x_1, x_2, x_3); y = \frac{x}{\varepsilon} \in Y_C = Y - C \right\} \tag{41}$$

And we assume that, there is no fissure intersecting the boundary $\partial\Omega$ of the open Ω . Introducing the following spaces:

$$V_\varepsilon^u = \{ u = (u_i); u_i \in H^1(\Omega_{C\varepsilon}); u_i|_{\partial\Omega} = 0 \} \tag{42}$$

$$V_\varepsilon^{*u} = \{ u = (u_i); u_i \in V_\varepsilon^u; [u_i N_i] \geq 0 \} \tag{43}$$

$$V_\varepsilon^\phi = \{ \phi; \phi \in H^1(\Omega_{C\varepsilon}); \phi|_{\partial\Omega} = 0 \} \tag{44}$$

$$V_\varepsilon^{*\phi} = \{ \phi; \phi \in V_\varepsilon^\phi; [\phi] \geq 0 \} \tag{45}$$

The corresponding variational formulation ($FVP\varepsilon$) of such a piezoelectric problem in $\Omega_{C\varepsilon}$, is then defined as follows:



Problem (FVP ε): Find $(u^\varepsilon, \phi^\varepsilon)$ in $V_\varepsilon^{*u} \times V_\varepsilon^{*\phi}$ such that:

$$\begin{cases} a(u, v - u^\varepsilon) + b(\phi, v - u^\varepsilon) \geq 0 & \forall v \in V_\varepsilon^{*u} \\ c(\phi, \varphi - \phi^\varepsilon) + d(u, \varphi - \phi^\varepsilon) \geq 0 & \forall \varphi \in V_\varepsilon^{*\phi} \end{cases} \quad (46)$$

In order to study the asymptotic behavior of the solution when ε tends to zero, we use the classical following expansions, both for the unknown and the test functions:

$$u^\varepsilon(x) = u^0(x) + \varepsilon u^1(x, y) + \varepsilon^2 u^2(x, y) + \dots \quad (47)$$

$$\phi^\varepsilon(x) = \phi^0(x) + \varepsilon \phi^1(x, y) + \varepsilon^2 \phi^2(x, y) + \dots \quad (48)$$

$$v^\varepsilon(x) = v^0(x) + \varepsilon v^1(x, y) + \varepsilon^2 v^2(x, y) + \dots \quad (49)$$

$$\varphi^\varepsilon(x) = \varphi^0(x) + \varepsilon \varphi^1(x, y) + \varepsilon^2 \varphi^2(x, y) + \dots \quad (50)$$

Introducing the following spaces:

$$V_{YC}^u = \{v = (v_i); v_i \in H^1(Y_C); Y - \text{periodic}\} \quad (51)$$

$$\tilde{V}_{YC}^u = \{v = (v_i); v \in V_{YC}^u; \tilde{v} = 0\} \quad (52)$$

$$V_{YC}^{*u} = \{v = (v_i); v \in V_{YC}^u; [v_i N_i] \geq 0 \text{ on } C\} \quad (53)$$

$$\tilde{V}_{YC}^{*u} = \{v = (v_i); v \in V_{YC}^{*u}; \tilde{v} = 0\} \quad (53)$$

$$V_{YC}^\phi = \{\varphi; \varphi \in H^1(Y_C); Y - \text{periodic}\} \quad (54)$$

$$\tilde{V}_{YC}^\phi = \{\varphi; \varphi \in V_{YC}^\phi; \tilde{\varphi} = 0\} \quad (55)$$

$$V_{YC}^{*\phi} = \{\varphi; \varphi \in V_{YC}^\phi; [\varphi] \geq 0\} \quad (56)$$

$$\tilde{V}_{YC}^{*\phi} = \{\varphi; \varphi \in V_{YC}^{*\phi}; \tilde{\varphi} = 0\} \quad (57)$$

Comparing (47)-(50) with (46), we get the following relations:

$$\begin{cases} \int_{\Omega} a_{ijkl} \frac{\partial u_k^0}{\partial x_l} \frac{\partial (v_j^0 - u_j^0)}{\partial x_i} dx + \int_{\Omega} a_{ijkl} \frac{\partial u_k^0}{\partial x_l} \left(\frac{\partial (v_j^1 - u_j^1)}{\partial y_i} \right)^{\sim} dx + \int_{\Omega} a_{ijkl} \left(\frac{\partial u_k^1}{\partial y_l} \right)^{\sim} \frac{\partial (v_j^0 - u_j^0)}{\partial x_i} dx \\ + \int_{\Omega} a_{ijkl} \left(\frac{\partial u_k^1}{\partial y_l} \frac{\partial (v_j^1 - u_j^1)}{\partial y_i} \right)^{\sim} dy + \int_{\Omega} e_{ijk} \frac{\partial \phi^0}{\partial x_k} \frac{\partial (v_i^0 - u_i^0)}{\partial x_j} dx + \int_{\Omega} e_{ijk} \frac{\partial \phi^0}{\partial x_k} \left(\frac{\partial (v_i^1 - u_i^1)}{\partial y_j} \right)^{\sim} dx \\ + \int_{\Omega} e_{ijk} \left(\frac{\partial \phi^1}{\partial y_k} \right)^{\sim} \frac{\partial (v_i^0 - u_i^0)}{\partial x_j} dx + \int_{\Omega} e_{ijk} \left(\frac{\partial \phi^1}{\partial y_k} \frac{\partial (v_i^1 - u_i^1)}{\partial y_j} \right)^{\sim} dy \geq 0; \forall v^0, v^1 \in (H_0^1(\Omega))^3 \end{cases} \quad (58)$$



$$\left\{ \begin{aligned} & \int_{\Omega} \varepsilon_{ij} \frac{\partial \phi^0}{\partial x_j} \frac{\partial(\phi^0 - \phi^1)}{\partial x_i} dx + \int_{\Omega} \varepsilon_{ij} \frac{\partial \phi^0}{\partial x_j} \left(\frac{\partial(\phi^1 - \phi^1)}{\partial y_i} \right)^{\sim} dx + \int_{\Omega} \varepsilon_{ij} \left(\frac{\partial \phi^1}{\partial y_j} \right)^{\sim} \left(\frac{\partial(\phi^0 - \phi^0)}{\partial x_i} \right) dx \\ & + \int_{\Omega} \varepsilon_{ij} \left(\frac{\partial \phi^1}{\partial y_j} \frac{\partial(\phi^1 - \phi^1)}{\partial y_i} \right)^{\sim} dy - \int_{\Omega} e_{ikl} \frac{\partial u_k^0}{\partial x_l} \frac{\partial(\phi^0 - \phi^0)}{\partial x_i} dx - \int_{\Omega} e_{ikl} \frac{\partial u_k^0}{\partial x_l} \left(\frac{\partial(\phi^1 - \phi^1)}{\partial y_i} \right)^{\sim} dx \\ & - \int_{\Omega} e_{ikl} \left(\frac{\partial u_k^1}{\partial y_l} \right)^{\sim} \frac{\partial(\phi^0 - \phi^0)}{\partial x_i} dx - \int_{\Omega} e_{ikl} \left(\frac{\partial u_k^1}{\partial y_l} \frac{\partial(\phi^1 - \phi^1)}{\partial y_i} \right)^{\sim} dy \geq 0; \forall \phi^0, \phi^1 \in H_0^1(\Omega) \end{aligned} \right. \quad (59)$$

with \sim represents the operator average defined on any Y-periodic function $f(y)$ by:

$$\tilde{f} = \frac{1}{|Y|} \int_Y f(y) dy \quad (60)$$

In the particular case where $v^1 = u^1$ and $\phi^1 = \phi^1$, we get:

$$\left\{ \begin{aligned} & \int_{\Omega} a_{ijkl} \left(\frac{\partial u_k^0}{\partial x_l} + \frac{\partial u_k^1}{\partial y_l} \right)^{\sim} \frac{\partial(v_j^0 - u_j^0)}{\partial x_i} dx + \int_{\Omega} e_{ijk} \left(\frac{\partial \phi^0}{\partial x_k} + \frac{\partial \phi^1}{\partial y_k} \right)^{\sim} \frac{\partial(v_i^0 - u_i^0)}{\partial x_j} dx = 0; \\ & \forall v^0, v^1 \in (H_0^1(\Omega))^3 \end{aligned} \right. \quad (61)$$

$$\left\{ \begin{aligned} & \int_{\Omega} \varepsilon_{ij} \left(\frac{\partial \phi^0}{\partial x_j} + \frac{\partial \phi^1}{\partial y_j} \right)^{\sim} \frac{\partial(\phi^0 - \phi^0)}{\partial x_i} dx - \int_{\Omega} e_{ikl} \left(\frac{\partial u_k^0}{\partial x_l} + \frac{\partial u_k^1}{\partial y_l} \right)^{\sim} \frac{\partial(\phi^0 - \phi^0)}{\partial x_i} dx = 0; \\ & \forall \phi^0, \phi^1 \in H_0^1(\Omega) \end{aligned} \right. \quad (62)$$

Consequently, the corresponding homogenized equations in which there are no more fissures then follow:

$$\left\{ \begin{aligned} & \frac{\partial \tilde{\sigma}_{ij}^0}{\partial x_j} = 0 \\ & \tilde{\sigma}_{ij}^0 = a_{ijkl} \left(\frac{\partial u_k^0}{\partial x_l} + \frac{\partial u_k^1}{\partial y_l} \right)^{\sim} + e_{ijk} \left(\frac{\partial \phi^0}{\partial x_k} + \frac{\partial \phi^1}{\partial y_k} \right)^{\sim} \end{aligned} \right. \quad (63)$$

and

$$\left\{ \begin{aligned} & \frac{\partial \tilde{D}_i^0}{\partial x_i} = 0 \\ & \tilde{D}_i^0 = \varepsilon_{ij} \left(\frac{\partial \phi^0}{\partial x_j} + \frac{\partial \phi^1}{\partial y_j} \right)^{\sim} - e_{ikl} \left(\frac{\partial u_k^0}{\partial x_l} + \frac{\partial u_k^1}{\partial y_l} \right)^{\sim} \end{aligned} \right. \quad (64)$$



HOMOGENIZED LOCAL PROBLEM IN THE PERIOD Y

Let us choose the fields v^1 and ϕ^1 as:

$$\begin{cases} v^1(x, y) - u^1(x, y) = \theta^u [w^u(y) - u^1(x, y)] \\ \theta^u \in D(\Omega); 0 \leq \theta^u \leq 1; w^u \in V_{YC}^{*u} \end{cases} \tag{65}$$

$$\begin{cases} \phi^1(x, y) - \phi^1(x, y) = \theta^\phi [w^\phi(y) - \phi^1(x, y)] \\ \theta^\phi \in D(\Omega); 0 \leq \theta^\phi \leq 1; w^\phi \in V_{YC}^{*\phi} \end{cases} \tag{66}$$

where $D(\Omega)$ represents the set of the infinitely derivable functions with compact support in Ω .

When we take (64) and (65) into account, then the comparison of (58) and (59) with (60) and (61) respectively, gives the following equations,

$$\begin{cases} \int_{\Omega} a_{ijkl} \left(\left(\frac{\partial u_k^0}{\partial x_l} + \frac{\partial u_k^1}{\partial y_l} \right) \frac{\partial (w_j^u - u_j^1)}{\partial y_i} \right)^{\sim} \theta^u dy + \int_{\Omega} e_{ijk} \left(\left(\frac{\partial \phi^0}{\partial x_k} + \frac{\partial \phi^1}{\partial y_k} \right) \frac{\partial (w_i^u - u_i^1)}{\partial y_j} \right)^{\sim} \theta^u dy \geq 0 \\ \forall \theta^u \in D(\Omega); 0 \leq \theta^u \leq 1 \end{cases} \tag{67}$$

$$\begin{cases} \int_{\Omega} \varepsilon_{ij} \left(\left(\frac{\partial \phi^0}{\partial x_j} + \frac{\partial \phi^1}{\partial y_j} \right) \frac{\partial (w^\phi - \phi^1)}{\partial y_i} \right)^{\sim} \theta^\phi dy - \int_{\Omega} e_{ikl} \left(\left(\frac{\partial u_k^0}{\partial x_l} + \frac{\partial u_k^1}{\partial y_l} \right) \frac{\partial (w^\phi - \phi^1)}{\partial y_i} \right)^{\sim} \theta^\phi dy \geq 0 \\ \forall \theta^\phi \in D(\Omega); 0 \leq \theta^\phi \leq 1 \end{cases} \tag{68}$$

Therefore, we locally obtain, respectively:

$$a_{ijkl} \left(\left(\frac{\partial u_k^0}{\partial x_l} + \frac{\partial u_k^1}{\partial y_l} \right) \frac{\partial (w_j^u - u_j^1)}{\partial y_i} \right)^{\sim} + e_{ijk} \left(\left(\frac{\partial \phi^0}{\partial x_k} + \frac{\partial \phi^1}{\partial y_k} \right) \frac{\partial (w_i^u - u_i^1)}{\partial y_j} \right)^{\sim} \geq 0; \forall w^u \in \tilde{V}_{YC}^{*u} \tag{69}$$

$$\varepsilon_{ij} \left(\left(\frac{\partial \phi^0}{\partial x_j} + \frac{\partial \phi^1}{\partial y_j} \right) \frac{\partial (w^\phi - \phi^1)}{\partial y_i} \right)^{\sim} - e_{ikl} \left(\left(\frac{\partial u_k^0}{\partial x_l} + \frac{\partial u_k^1}{\partial y_l} \right) \frac{\partial (w^\phi - \phi^1)}{\partial y_i} \right)^{\sim} \geq 0; \forall w^\phi \in \tilde{V}_{YC}^{*\phi} \tag{70}$$

and the local homogenized problem (**LHP**) in Y, then follows:

Problem (LHP): Find (u^1, ϕ^1) in $\tilde{V}_{YC}^{*u} \times \tilde{V}_{YC}^{*\phi}$ such that we obtain, for given $u^0(x)$ and $\phi^0(x)$:

$$\begin{cases} \int_Y a_{ijkl} \left(\left(\frac{\partial u_k^0}{\partial x_l} + \frac{\partial u_k^1}{\partial y_l} \right) \frac{\partial (w_j^u - u_j^1)}{\partial y_i} \right) dy + \int_Y e_{ijk} \left(\left(\frac{\partial \phi^0}{\partial x_k} + \frac{\partial \phi^1}{\partial y_k} \right) \frac{\partial (w_i^u - u_i^1)}{\partial y_j} \right) dy \geq 0 \\ \forall w^u \in \tilde{V}_{YC}^{*u} \end{cases} \tag{71}$$

and



$$\left\{ \begin{array}{l} \int_Y \varepsilon_{ij} \left(\left(\frac{\partial \phi^0}{\partial x_j} + \frac{\partial \phi^1}{\partial y_j} \right) \frac{\partial (w^\phi - \phi^1)}{\partial y_i} \right) dy - \int_Y e_{ikl} \left(\left(\frac{\partial u_k^0}{\partial x_l} + \frac{\partial u_k^1}{\partial y_l} \right) \frac{\partial (w^\phi - \phi^1)}{\partial y_i} \right) dy \geq 0 \\ \forall w^\phi \in \tilde{V}_{YC}^{*\phi} \end{array} \right. \quad (72)$$

All these results can then be summarized through the following proposition:

Proposition3. Under the expansions (47) and (48) for the solution $(u^\varepsilon(x, y), \phi^\varepsilon(x, y))$ of Problem (46), the first term $(u^0(x), \phi^0(x))$ satisfies Eqs. (62)-(63) and appropriate boundary conditions. Furthermore, for given $(u^0(x), \phi^0(x))$, the field $(u^1(x, y), \phi^1(x, y))$ is the solution of the nonlinear problem (LHP; Eqs. 70-71), and $(\tilde{\sigma}_{ij}^0, \tilde{D}_i^0)$ is therefore, defined as functions of $grad_x(u^0(x))$ and $grad_x(\phi^0(x))$. So, Eqs. (70)-(71) represent a nonlinear piezoelectric law.

ANALYSIS OF THE (STRAIN, ELECTRIC POTENTIAL)-(STRESS, ELECTRIC DISPLACEMENT) LAW

Remark1. Problem (LHP) can be written as in the following simplified form:

Find (u^1, ϕ^1) in $\tilde{V}_{YC}^{*u} \times \tilde{V}_{YC}^{*\phi}$ such that we obtain, for given $u^0(x)$ and $\phi^0(x)$:

$$\left\{ \begin{array}{l} \int_Y a_{ijkl} (h_{kix}(u^0) + h_{kly}(u^1)) b_{jij}(w^u - u^1) dy \\ \quad + \int_Y e_{ijk} (h_{kix}(\phi^0) + h_{kly}(\phi^1)) b_{jij}(w^u - u^1) dy \geq 0 \\ h_{kix}(\cdot) = \frac{\partial(\cdot)_k}{\partial x_l}; h_{kly}(\cdot) = \frac{\partial(\cdot)_k}{\partial y_l}; h_{kix}(\cdot) = \frac{\partial(\cdot)}{\partial x_l} \\ \forall w^u \in \tilde{V}_{YC}^{*u} \end{array} \right. \quad (73)$$

and

$$\left\{ \begin{array}{l} \int_Y \varepsilon_{ij} (h_{jix}(\phi^0) + h_{jly}(\phi^1)) b_{jij}(w^\phi - \phi^1) dy - \int_Y e_{ikl} (h_{kix}(u^0) + h_{kly}(u^1)) b_{jij}(w^\phi - \phi^1) dy \geq 0 \\ h_{jly}(\cdot) = \frac{\partial(\cdot)}{\partial y_k} \\ \forall w^\phi \in \tilde{V}_{YC}^{*\phi} \end{array} \right. \quad (74)$$

Remark2. Denoting

$$H_{kl} = h_{kix}(u^0); h_{kl} = h_{kly}(u^1); u = u^1; \phi = \phi^1; \sigma_{ij} = \sigma_{ij}^0; D_i = D_i^0 \quad (75)$$

Problem (LHP) can then be formulated as follows:

Find (u, ϕ) in $\tilde{V}_{YC}^{*u} \times \tilde{V}_{YC}^{*\phi}$ such that we obtain, for given $H_{kl} \in \mathbf{R}^6$ and $H_k \in \mathbf{R}^3$:

$$\left\{ \begin{array}{l} \int_Y a_{ijkl} (H_{kl} + h_{kl}(u)) b_{jij}(w^u - u) dy + \int_Y e_{ijk} (H_k + h_k(\phi)) b_{jij}(w^u - u) dy \geq 0 \\ \forall w^u \in \tilde{V}_{YC}^{*u} \end{array} \right. \quad (76)$$



and

$$\begin{cases} \int_Y \varepsilon_{ij} (H_j + h_j(\phi)) b_i(w^\phi - \phi) dy - \int_Y e_{ikl} (H_{kl} + b_{kl}(u)) b_i(w^\phi - \phi) dy \geq 0 \\ \forall w^\phi \in \tilde{V}_{YC}^* \phi \end{cases} \quad (77)$$

Therefore, $\sigma_{ij} = \sigma_{ij}^0$ and $D_i = D_i^0$ can be written as follows:

$$\begin{cases} \sigma_{ij} = a_{ijkl} (H_{kl} + b_{kl}(u)) + e_{ijk} (H_k + b_k(\phi)) \\ D_i = \varepsilon_{ij} (H_j + h_j(\phi)) - e_{ikl} (H_{kl} + b_{kl}(u)) \end{cases} \quad (78)$$

So, the homogenized (strain, electric potential)-(stress, electric displacement) law is characterized by the function defined by:

$$(H_{kl}, b_{kl}) \mapsto (\tilde{\sigma}_{ij}, \tilde{D}_i) \quad (79)$$

Nevertheless, for the study of (79), let us introduce the following functions, defined from $\mathbf{R}^6 \times \mathbf{R}^3$ towards \mathbf{R} , by:

$$\begin{aligned} W(H_{sm}, H_s) &= \frac{1}{2|Y|} \int_Y a_{ijkl} (H_{ij} + h_{ij}(u)) (H_{lm} + b_{lm}(u)) dy \\ &\quad + \frac{1}{2|Y|} \int_Y e_{ijk} (H_i + b_i(\phi)) (H_{jk} + b_{jk}(u)) dy \end{aligned} \quad (80)$$

$$\begin{aligned} W^*(H_{sm}, H_s) &= \frac{1}{2|Y|} \int_Y \varepsilon_{ij} (H_i + b_i(\phi)) (H_j + h_j(\phi)) dy \\ &\quad - \frac{1}{2|Y|} \int_Y e_{ikl} (H_{ik} + b_{ik}(u)) (H_l + b_l(\phi)) dy \end{aligned} \quad (81)$$

Moreover, the proposition that follows presents the main result of this analysis:

Proposition 4. The functions defined above, through (80) and (81), are of class C^1 , positive; $\tilde{\sigma}$ and \tilde{D} satisfying the following relations:

$$\begin{cases} \delta W \geq \frac{1}{2} \tilde{\sigma}_{ij} \delta H_{ij} \\ \delta W^* \geq \frac{1}{2} \tilde{D}_i \delta H_i \end{cases} \quad (82)$$

Proof. As u and ϕ are continuous functions of $\underline{H} = (H_{ij})_{i,j=1,2,3}$ and $\underline{H} = (H_i)_{i=1,2,3}$, defined from \mathbf{R}^6 (resp. \mathbf{R}^3) towards V_{YC}^u (resp. V_{YC}^ϕ), W and W^* are then of class C^0 . Let us now introduce:

$$\begin{cases} h_{ij}^* = h_{ij}(u) + H_{ij} \\ \text{and} \\ h_i^* = h_i(u) + H_i \end{cases} \quad (83)$$



So, by virtue of (3)-(5), we can formulate the variation of W and W^* as follows:

$$\begin{aligned} \delta W &= \frac{1}{|Y|} \int_Y a_{ijk} h_{ij}^* \delta h_{kl}^* dy + \frac{1}{2|Y|} \int_Y a_{ijkl} \delta h_{ij}^* \delta h_{kl}^* dy + \frac{1}{2|Y|} \int_Y e_{ijk} \delta h_i^* \delta h_{jk}^* dy \\ &\quad + \frac{1}{2|Y|} \int_Y e_{ijk} \delta h_i^* h_{jk}^* dy + \frac{1}{2|Y|} \int_Y e_{ijk} h_i^* \delta h_{jk}^* dy \\ &\geq \frac{1}{|Y|} \int_Y a_{ijkl} h_{ij}^* \delta h_{kl}^* dy + \frac{1}{2|Y|} \int_Y a_{ijkl} h_{ij}^* \delta H_{kl} dy + \frac{1}{2|Y|} \int_Y e_{ijk} h_i^* \delta h_{jk}^* dy \\ &\quad + \frac{1}{2} \sigma_{kl}^- \delta H_{kl} \end{aligned} \tag{84}$$

$$\begin{aligned} \delta W^* &= \frac{1}{|Y|} \int_Y \varepsilon_{ij} h_i^* \delta h_j^* dy + \frac{1}{2|Y|} \int_Y \varepsilon_{ij} \delta h_i^* \delta h_j^* dy - \frac{1}{2|Y|} \int_Y e_{ikl} \delta h_{ik}^* h_l^* dy \\ &\quad - \frac{1}{2|Y|} \int_Y e_{ikl} h_{ik}^* \delta h_l^* dy - \frac{1}{2|Y|} \int_Y e_{ikl} \delta h_{ik}^* \delta h_l^* dy \\ &\geq \frac{1}{2|Y|} \int_Y \varepsilon_{ij} \delta h_i^* \delta h_j^* dy - \frac{1}{2|Y|} \int_Y e_{ikl} \delta h_{ik}^* \delta h_l^* dy + \frac{1}{2} D_j^- \delta h_j \end{aligned} \tag{85}$$

Taking (76) and (77) into account, we get:

$$\begin{cases} \delta W \geq \frac{1}{2} \sigma_{ij}^- \delta H_{kl} \\ \delta W \geq \frac{1}{2} D_i^- \delta H_i \end{cases} \tag{86}$$

CONCLUSION

From the variational formulation of the three-dimensional problem of Linear Piezoelectricity, we deduced that corresponding to a cracked piezoelectric structure. Considering afterward the case of a structure presenting a periodic distribution of cracks, we managed to build, on the homogenization period, the homogenized formulation of the corresponding problem, as a result of an asymptotic development of the solution. A non-linear law between the mechanical strain and the electric potential on one hand, and the mechanical stress and the electric displacement on the other hand, has been then established.

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