Analytical evaluation of J-integral for an elliptical notch

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ABSTRACT. *In the present paper the J-integral for an elliptical notch has been analytically calculated by using the stress distribution for the case of an isolated ellipse under remote tensile loading. The material is thought of as obeying a purely linear elastic law and the difference between the J-integral of a crack and the equivalent ellipse is discussed. For instance it is analytically verified that when the notch tip radius tends toward zero the well known J-integral formula for a crack is obtained. Finally, as an engineering application, an accurate formula is given for the evaluation of the Notch Stress Intensity Factors of a crack from the peak stress of an equivalent ellipse.*

1. INTRODUCTION

The path integral proposed by Rice [1] was defined in order to measure the stress field intensity near a sharp notch. By using Irwin's equations, Rice evaluated the J-integral for a crack far from the tip without taking into account the actual crack tip shape. It's well known that the J-integral is path independent and is related only to relevant Stress Intensity Factors (SIF). On the basis of a reasonable assumption, Rice postulated that the J-integral applied to sharp notch equals the J-integral related to crack with the same length. However, he evaluated the J-integral for a rounded notch with low accuracy, and the consequent relationship between the peak stress and the SIF was approximated.

On the basis of a new analytical equation for tangential stress on the free surface, in ref [2] a high order of accuracy for J-integral applied to U-notch with linear flank was obtained. Furthermore, when the notch depth-to-radius ratio tends toward infinity, many numerical analyses confirmed the overlapping of J-integral for the two type of geometric discontinuity [2].

As far as non-linear materials are concerned, analogous results to elastic material may be obtained. Rice, and more recently Matvienko and Morozov [3], gave some approximate expression for the maximum intensity of strain and stress on the surface of the notch tip using the radius of curvature ρ, the mechanical proprieties of the material and the J-integral.

With regard to opening V-notches, Hasebe and Iida [4] expanded the relation between the Notch Stress Intensity Factors (N-SIF) and the peak in terms of a power series in ρ. For the sake of simplicity, a linear relation was also written from N-SIF and peak stress where the proportional coefficient was estimated as a function of the opening angle 2α. Similar results were also obtained in reference [2] by taking into account the J-integral applied to V-notches after the definition of a new path-independent operator.

The aim of this paper is to analytically evaluate the J-integral for an elliptical hole in a wide plate subject to tension. So that, when the tip radius tends to zero, an analytical proof is obtained for the overlapping of J-integral of the ellipse and J-integral of an equivalent crack. Additionally, a new accurate equation is obtained for the evaluation of peak stress from SIF and a comparison with many numerical data taken from the literature has been made.

2. NORMAL STRESS COMPONENTS ON THE ELIPTIC HOLE IN A PLATE UNDER TENSION

The general stress problem of an elliptical hole in a linear-elastic plate was studied by Inglis. In this paper, we take into account only the special case of an ellipse in a wide plate under tension subjected to remote traction loading σ_{nom} as reported in figure 1. The tangential stress σ_t, along the ellipse border assumes a simple analytical formulation (see Timoshenko and Goodier [5]):

$$\sigma_t = \sigma_{nom}\ e^{2\xi_0}\left[\frac{(1+e^{-2\xi_0})\ \sinh 2\xi_0}{\cosh 2\xi_0 - \cos 2\eta} - 1\right] \tag{1}$$

where

$$\xi_0 = \operatorname{arcth}\frac{b}{a} \tag{2}$$

a and *b* being the two semiaxes of the ellipse while η is the elliptical angle defined as:

$$x = a\ \cos\eta \qquad y = b\ \sin\eta \tag{3}$$

As well know, the greatest value occurring at the ends of the major axis reduces to

$$\sigma_{t,max} = \sigma_{nom}\left(1+2\frac{a}{b}\right) \tag{4}$$

This increases without limit when the hole becomes more and more slender. In fact, for the limit case of crack only the SIF is defined.
In order to analytically evaluate the J-Integral for an ellipse, eq. (1) may be arranged as functions of *y* co-ordinate. After some manipulations eq. (1) becomes:

$$\sigma_t = \sigma_{nom}\ \frac{c^2}{(a-b)^2}\left[\frac{a\,b}{c^2}\ \frac{1+\frac{(a-b)^2}{c^2}}{\frac{b^2}{c^2}+\frac{y^2}{b^2}}\ -\ 1\right] \tag{5}$$

where *c* is defined as

$$c = \sqrt{a^2 - b^2} \tag{6}$$

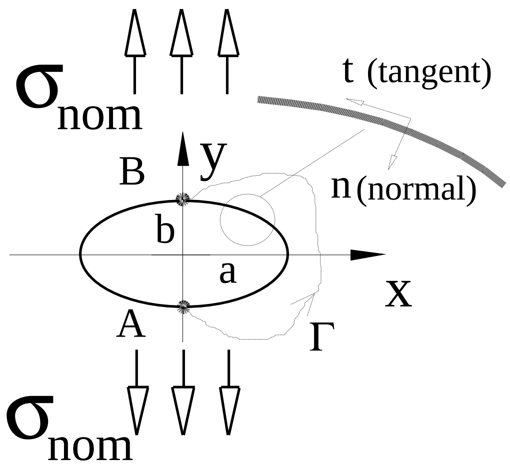


Figure 1. Ellipse in a wide plate under tension

3. ANALITYCAL EVALUATION OF J-INTEGRAL FOR AN ELIPTIC HOLE IN A PLATE UNDER TENSION

The evaluation of J-integral on a inner path Γ from the two opposite points A and B of figure 1, here called J_{el}, is equivalent to a surface integral along the ellipse border. J_{el} assumes the form:

$$J_{el} = \int_0^b \frac{\sigma_t^{\;2}}{E'}\, dy \tag{7}$$

where E' is equal to the Young modulus E for plane stress or to $E/(1-\nu^2)$ for plane strain, respectively.
By means of eq. (5) the J_{el} integral becomes:

$$J_{el} = \frac{\sigma_{nom}^{\;2}}{E'} \int_0^b \left\{ \frac{b^6\, a^2}{(a-b)^4} \quad \frac{\left[c^2 + (a-b)^2\right]^2}{\left(b^4 + c^2\, y^2\right)^2} - 2\,a\,c^2\, b^3\, \frac{c^2 + (a-b)^2}{(a-b)^4}\, \frac{1}{b^4 + c^2\, y^2} + \frac{c^4}{(a-b)^4} \right\} dy =$$

$$= \frac{\sigma_{nom}^{\;2}}{E'} \left\{ \frac{2\,a^4}{c\,(a-b)^2} \left[\frac{c\,b}{a^2} + \operatorname{arctg}\frac{c}{b} \right] - 4\, \frac{a^2\, c\;\; b}{(a-b)^3}\; \operatorname{arctg}\frac{c}{b} + \frac{b\,c^4}{(a-b)^4} \right\} \tag{8}$$

Figure 2 shows the trend of J_{el} against the minor axis *b*. The evaluation of the integral on the r.h.s. in eq. (8) is very standard. In spite of the articulate form of eq. (8), the trend of J_{el} is close to a linear behaviour. By means of eq. (4) we may introduce the peak stress $\sigma_{t,max}$ into eq. (8):

$$J_{el} = \frac{\sigma_{t,max}^{\;\;2}}{\left(1 + 2\,\frac{a}{b}\right) E'} \left\{ \frac{2\,a^4}{c\,(a-b)^2} \left[\frac{c\,b}{a^2} + \operatorname{arctg}\frac{c}{b} \right] - 4\, \frac{a^2\, c\;\; b}{(a-b)^3}\; \operatorname{arctg}\frac{c}{b} + \frac{b\,c^4}{(a-b)^4} \right\} \tag{9}$$

If we consider the special case of a circle, from eq. (8) we have:

$$J_c = \lim_{b \to a} J_{el} = \frac{21}{5}\,\frac{\rho}{E'}\,\sigma_{nom}^{\;2} \tag{10}$$

being ρ the radius of the circle.
As well know, for a crack of length 2*a* in a wide plate under tension, the J-integral assumes the form:

$$J = \frac{K_I^{\;2}}{E'} = \frac{\sigma_{nom}^{\;2}\,\pi\,a}{E'} \tag{11}$$

where K_I is the SIF for mode I loadings. Therefore, the equation of the straight line between the two limiting cases of crack and of the circle becomes:

$$J = \frac{\sigma_{nom}^{\;2}\,\pi\,a}{E'} \left[\frac{b}{a} \left(\frac{21}{5\,\pi} - 1 \right) + 1 \right] \tag{12}$$

Eq.(12) is reported in figure 2 as "linear behaviour".
Finally, the limits of eq. (9) when b tends towards zero, is given by:

$$J_c = \lim_{b\to 0} J_{el} = \lim_{b\to 0} \frac{\sigma_{nom}^2}{E'} \frac{2a^4}{c(a-b)^2}\left(\operatorname{arctg}\frac{c}{b}\right) = \frac{\sigma_{nom}^2 \pi a}{E'} \tag{13}$$

(for $b\to 0$, and therefore $c\to a$).

Eq. (13) demonstrates that the J-integral for a crack is exactly the limit of J_c for an ellipse when the minor axis tends towards zero, so that Rice's hypothesis is analytically confirmed in the case of an ellipse.

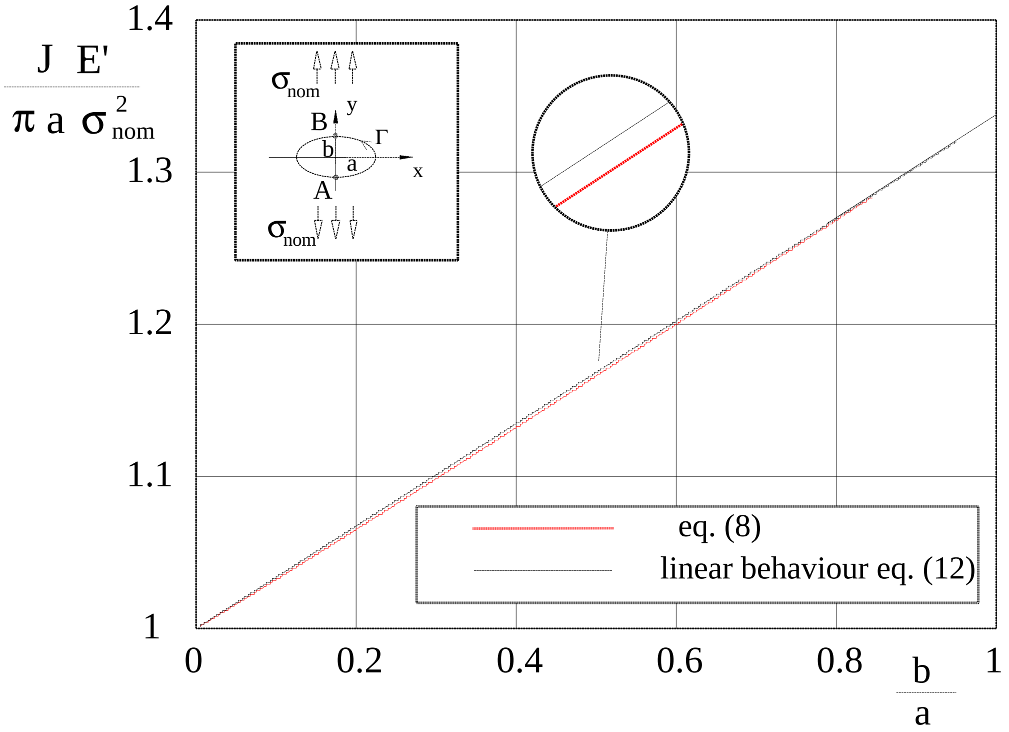


Figure 2. Comparison between the J-integral of an ellipse and that of the equivalent crack

4. PEAK STRESS FROM SIF

From an engineering point of view, the peak stress $\sigma_{t,max}$ may be calculated on the basis of the results of figure 2. By comparing eq. (9) and eq. (11) we have:

$$\frac{K_I^2}{E'} = \frac{J_{el}}{w(a/b)} \tag{14}$$

where *w* is the weight correction function obtained from the former case of an ellipse in an infinite plate:

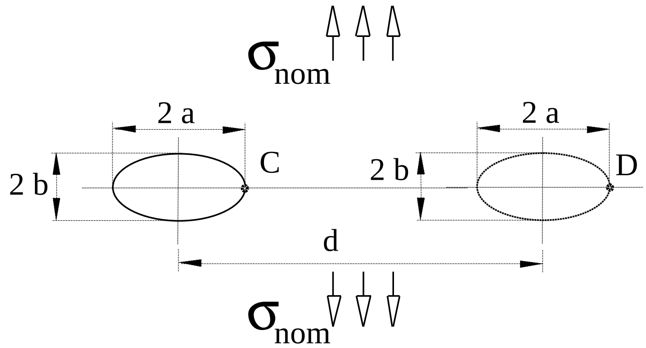
$$w = \frac{b}{a}\left(\frac{21}{5\pi} - 1\right) + 1 \tag{15}$$

J_{el} in eq. (14) is evaluated by means of eq. (9). So that, a proper relationship was proposed from peak stress and the SIF. Tables 1 and 2 report values of peak stress evaluated from the relative SIF by using of eq. (14). All numerical data are taken from the work of Shin at al. [6] where the difference from FEA and the Inglis' exact solution was found to be around 1-2% when the *a/b* ratio range from 0.107 to 2.979.

Tables 1 and 2 show a sound agreement between the FE results and the analytical ones. The average error is about 2% and it takes into account the prospective error in SIF evaluation.

For equivalent crack

$K_I = Y \sigma_{nom} \sqrt{\pi a}$

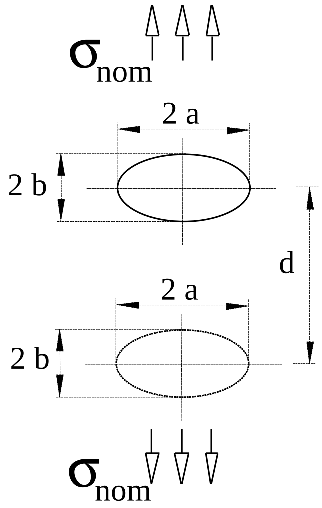


	2a/d	a [mm]	b [mm]	b/a	Y*	$\sigma_{t,max}$ eq. (14) [MPa]	$\sigma_{t,max}$ FEA [MPa]	Error [%]
C (inner edge)	0.7	70	7.5	0.107	1.132	22.28	22.62	-1.51
	0.7	70	10	0.143	1.132	17.00	16.97	0.15
	0.7	70	14	0.200	1.132	12.47	12.28	1.51
	0.7	70	23.5	0.336	1.132	7.89	7.96	-0.94
	0.8	70	23.5	0.336	1.225	8.53	8.68	-1.69
D (outer edge)	0.7	70	7.5	0.107	1.058	20.82	20.75	0.35
	0.7	70	10	0.143	1.058	15.88	15.82	0.41
	0.7	70	14	0.200	1.058	11.65	11.58	0.61
	0.7	70	23.5	0.336	1.058	7.37	7.30	0.96
	0.8	70	23.5	0.336	1.081	7.53	7.44	1.21

Table 1. Comparison between the predicted and FEA peak stresses [6] for two identical collinear ellipses in an infinite plate aligned perpendicular to the axial loading direction (σ_{nom}=1 MPa)
* Dimensionless shape factor from [7] reported in [6]

For equivalent crack

$$K_I = Y \sigma_{nom} \sqrt{\pi a}$$



2a/d [mm]	a [mm]	b [mm]	b/a	Y*	$\sigma_{t,max}$ eq (14) [MPa]	$\sigma_{t,max}$ FEA [MPa]	Error [%]
1.17	70	7.5	0.107	0.813	16.00	16.55	-3.32
1.17	70	10	0.143	0.813	12.21	12.54	-2.67
1.17	70	14	0.200	0.813	8.95	9.11	-1.73
1.17	70	23.5	0.336	0.813	5.66	5.97	-5.14

Table 2. Comparison between the predicted and FEA peak stresses [6] for two identical collinear ellipses in an infinite plate aligned in the axial loading direction (σ_{nom} =1 MPa)
* Dimensionless shape factor from [6] reported in [6]

5. CONCLUSIONS

In this paper it has been established, for an ellipse in a wide plate under tension of semiaxis *a-b*, the analytical relationship of J-integral. The difference, in terms of J-integral, between the ellipse and the equivalent crack is a linear function of the minor axis *b* of the ellipse. So that, under the hypothesis that for general geometry the difference depends only from the *a/b* ratio, a general relationship from the peak stress of the ellipse and the Stress Intensity Factors of an equivalent crack is obtained. By taking into account numerical analysis from literature, the peak stress is evaluated with an average error of 1.6% with *b/a* ratio ranging from 0.11 to 0.34.

6. REFERENCES

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