CONSTITUTIVE MODELING FOR THE UNSTABLE STATES OF MATERIALS BY WAVE DYNAMICS AND VARIATIONAL METHODS

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ABSTRACT
When fracture happens, in several problems of engineering the selection of the constitutive relation has an essential role beyond the equations of motion and the kinematical equation. A wide range of studies published in this topic use various types of constitutive models being more or less appropriate approximations to the mechanical problem under consideration. In searching for a generalized approach our paper deals with the theory of constitutive modeling of solid materials. The studies presented int he following are based on the almos obvious concept that the set of basic equations descibing solid bodies should possess mathemetical consistency. In this work two basic reqiurements are considered. The one is the existence and regular propagation of waves, while the other is a so-called generic behavior at material instability. The first condition is of dynamic nature appearing at rapid, high rate loading. The second one is closely related to fracture: we assume that the onset of material instability is a starting point of the process, which leads to fracture. The methods we use are based on variational principles and the theory of dynamical systems. For this reason we should define an infinite dimensional dynamical system, which is determined by the set of fundamental equations of the solid body. Such equations are used to define differential opeartors acting on the basic field variables like displacement, velocity, stress and strain, which satisfy appropriate boundary conditions. Then we concentrate on a selected state S^0 of the material, which satisfies all the basic equations and additionally all the bondary conditions. Now state S^0 of the material is called stable, if the solution of the dynamical system satisfies the conditions of the Liapunov stability. By using the methods of the theory of dynamical systems we will study how material instability happens and we judge whether it is a generic type of loss of stability or not.

1 INTRODUCTION

Fracture phenomena are initiated by some kind of loss of stability (Drucker [1], Rice [2], Rice et al. [3], [4]) of the material. When a mathematical description is needet to do an apriori calculation we should form a set of basic equations (Lubliner [5]) to perform numerical analysis (De Borst et al. [6], Zbib, Aifantis [7]).

The essential role is of the constitutive equations describing material properties. There are several ideas to build such equations. In this paper we follow Mindlin's theory and require regular wave dynamical properties.

2 THE CONSTITUTIVE EQUATON

For determinig stress σ Mindlin's idea was to use variational identity

$$\delta\int_V\int_{t_0}^{t_1} u \, dtdV \equiv \delta\int_V\int_{t_0}^{t_1} \sigma\cdot\cdot\delta\varepsilon \, ,$$

where u, ε are displacement and strain. Additionally we require that the system of fundamental equations built by using the constitutive equation should be available to determine an acceleration wave with finite speed of propagation. Assume that the constitutive equation contains stress, strain and the first derivatives of them. By using internal variable $\eta(\varepsilon)$ we have $u = (\varepsilon, \sigma, \eta)$. Then Mindlin's variational form leads to the Lagrange derivative of u, which contains the second derivative $\frac{\partial^2 \varepsilon}{\partial \varepsilon_t \partial \varepsilon_t}$, where $\varepsilon_t = \frac{\partial \varepsilon}{\partial t}$. Assume that the selection of the constitutive equation excludes it. Then

$$u = F(\sigma, \eta, \varepsilon)\varepsilon_t + G(\sigma, \varepsilon),$$

from which

$$\sigma = \frac{\partial G}{\partial \varepsilon} - \frac{\partial F}{\partial \sigma}\sigma_t - \frac{\partial F}{\partial \eta}\frac{\partial \eta}{\partial \varepsilon}\varepsilon_t \cdot$$

There exists an acceleration wave with finite wave speed, if

$$-B(\sigma, \varepsilon)\frac{\partial(\,)}{\partial \sigma_t} - \frac{\partial(\,)}{\partial \varepsilon_t} = 0 ,$$

where B is a positive function and notation

$$(\,) = \frac{\partial G}{\partial \varepsilon} - \frac{\partial F}{\partial \sigma}\sigma_t - \frac{\partial F}{\partial \eta}\frac{\partial \eta}{\partial \varepsilon}\varepsilon_t$$

is used. Then

$$B\frac{\partial F}{\partial \sigma} + \frac{\partial F}{\partial \eta}\frac{\partial \eta}{\partial \varepsilon} = 0$$

thus

$$\frac{\partial F}{\partial \eta}\frac{\partial \eta}{\partial \varepsilon} = -B\frac{\partial F}{\partial \sigma} \cdot$$

When the internal variable is the stress, the constitutive equation is

$$\sigma = \frac{\partial G}{\partial \varepsilon} - \frac{\partial F}{\partial \sigma}(\sigma_t - B\varepsilon_t) ,$$

or in general form

$$F_0(\sigma, \varepsilon) + F_1(\sigma, \varepsilon)\frac{\partial \sigma}{\partial t} + F_2(\sigma, \varepsilon)\frac{\partial \varepsilon}{\partial t} = 0 \cdot$$

3 DYNAMICAL SYSTEMS

While stability is considered to be a local property, we may add infinitesimal perturbations to the basic variables and use small deformation theory for the set of basic equations written for the field of perturbations. For the sake of simplicity the study is restricted to uniaxial problems, but a quite similar approach is possible even for a general three-axial case.

We may denote as (small) perturbation variables of state S^0 by v, ε, σ for the fields of velocity, strain and the symmetric stress. Then the basic equations are

$$\dot{\varepsilon} = \frac{1}{2}\left(\dot{v} \circ \nabla + \nabla \circ \dot{v}\right), \tag{1}$$

$$\rho \dot{v} = \sigma \nabla , \tag{2}$$

$$F_0(\sigma, \varepsilon) + F_1(\sigma, \varepsilon)\frac{\partial \sigma}{\partial t} + F_2(\sigma, \varepsilon)\frac{\partial \varepsilon}{\partial t} = 0, \tag{3}$$

where ρ is mass density as usual.

To the set of perturbation field equations we should add homogeneous boundary conditions and the question of stability means an investigation of the behavior of the boundary value problem formed by the set of equations eqns (1), (2) and (3).

For performing a stability analysis let us define a dynamical system of infinite dimension (Temam [8]). For this reason we assume that equations eqns (1), (2) and (3) can be transformed into the perturbation velocity field v satisfying homogeneous boundary conditions. Then

$$\rho v_{tt} = \left(-(F_2)^{-1} F_1 \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\left((F_2)^{-1} F_1\right)\right) v + \frac{\partial (F_2)^{-1}}{\partial x} F_0 \tag{4}$$

is obtained. Assume that

$$\frac{\partial (F_2)^{-1}}{\partial x} F_0 = 0.$$

Now an infinite dimensional dynamical system

$$\ddot{v} = F^1 v + F^2 \dot{v} \tag{5}$$

is defined by eqn (5), where (differential) operators $F^i, i = 1,2$ acting in field v are defined. Then by introducing new variables

$$y_1 = v,$$
$$y_2 = \dot{v}$$

a formal mathematical transformation may result a first order abstract dynamical system

$\dot{y}_1 = y_2,$ (6)

$\dot{y}_2 = F^1 y_1 + F^2 y_2,$ (7)

By applying Lyapunov's indirect method (Troger, Steindl [9]) we should derive the characteristical equation of the system of eqns (6), (7)

$$\lambda y_1 = y_2, \qquad \lambda y_2 = F^1 y_1 + F^2 y_2. \tag{8}$$

After proper rearrangements eqn (8) has the form

$$\lambda^2 y_1 - \lambda F^2 y_1 - F^1 y_1 = 0 \tag{9}$$

and the stability condition is

- $\operatorname{Re}\lambda_i < 0$, i=1... for all λ_i satisfying eqn (9).

The stability boundary (a loss of stability may be possible) is

- there exists a critical λ_{cr}, for which $\operatorname{Re}\lambda_{cr} = 0$, while $\operatorname{Re}\lambda_i < 0$, for all $\lambda_i \neq \lambda_{cr}$ satisfying eqn (9).

Two possible generic ways of loss of stability are called

- the *static bifurcation (SB),* when

$\operatorname{Re}\lambda_{cr} = 0$; (10)

- the *dynamic bifurcation (DB),* when

$\operatorname{Re}\lambda_{cr} \neq 0$. (11)

Then from eqns (9) and (10) the (necessary) condition of the static bifurcation type of material instability is

$$F^1 y_1 = 0, \tag{12}$$

while eqns (9) and (11) imply the (necessary) condition for the dynamic bifurcation

$$F^2 y_1 = 0. \tag{13}$$

Note that conditions eqns (12) and (13) are partial differential equations with homogeneous boundary conditions. Having done all the necessary substitutions and rearrangements the condition of eqn (12) implies

$$\left(-(F_2)^{-1} F_1 \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\left((F_2)^{-1} F_1\right)\right) v = 0 \tag{14}$$

and eqn (13) is satisfied automatically (and trivially)

$$F^2 v \equiv 0. \tag{15}$$

From eqn (15) we see that dynamic bifurcation or the flutter case (Bigoni, Willis [10]) does not appear at this material.

Remark that the same bifurcation conditions (eqns (12) and (13)) can be obtained for the three axial case. Unfortunately, in such general case the implied boundary value problem is a very complicated one and no analytic method can be found to evaluate it. Then we could do a more or less restrictive weak formulation or search for some numerical method.

4 SUMMARY

When Mindlin's idea and the basic wave dynamical requirements are applied a general form is obtained for the constitutive equation. To use such formulation in material instability cases we should add condition $\frac{\partial (F_2)^{-1}}{\partial x} F_0 = 0.$ Then we have a constitutive equations which can be used for calculations in various localization type instability phenomena (like micro or macro scale shear band formulation for example).

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